High-Order/hp-Adaptive Discontinuous Galerkin Finite Element Methods for Compressible Fluid Flows

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Joint work with
Paul Houston (Nottingham)

www.AptoFEM.com

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Outline

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1. **Introduction**
   - Framework
2. **Adaptive algorithms**
   - h–Adaptivity
   - p–Adaptivity

Abstract

Aims

1. Construction of *High–Order DGFEMs* for a class of second–order (quasilinear) PDEs;
2. Develop the *a posteriori* error analysis and adaptive mesh design of the DGFEM approximation of target functionals of the solution based on employing *anisotropic* h–/hp–refined meshes.

Measurement Problem for Compressible

- Measurement Problem: Given a user–defined tolerance $\texttt{TOL} > 0$, can we efficiently design $S_{h,p}$ such that

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \texttt{TOL}.$$

Fluid dynamics:	drag and lift coefficients.
Other examples:	point value, flux, mean value, etc.

- Applications
  - Compressible (aerodynamic) flows. ADIGMA

P. Houston (Nottingham), R. Hartmann (DLR, Braunschweig)

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1. Introduction
   - Framework
2. **Adaptive algorithms**
   - h–Adaptivity
   - p–Adaptivity

- Goal:

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} |\eta_\kappa(u_h)| \leq \texttt{Tol}.$$

- Automatic refinement algorithm:
  0. Start with initial (coarse) grid $\mathcal{T}_h^{(j=0)}$.
  1. Compute the numerical solution $u_h^{(j)}$ on $\mathcal{T}_h^{(j)}$.
  2. Compute the local error indicators η_κ.

Adaptivity

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Adaptive Algorithm (*hp*–Adaptivity)

- Regularity estimation via truncated Legendre series expansions.
  Houston, Senior & Süli 2003, Houston & Süli 2005, Eibner & Melenk 2005.
- If both u and z are deemed to be non-smooth, apply anisotropic h–refinement.
- Else, perform anisotropic p–refinement

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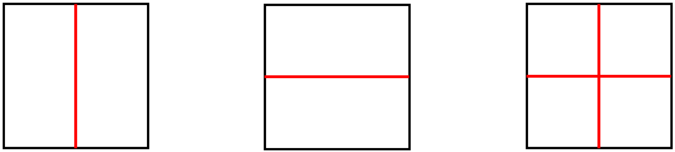
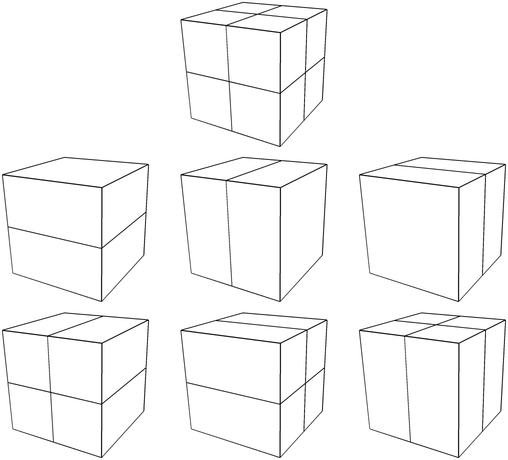
Adaptive Algorithm (h–Adaptivity)

- Element refinements.



$\mathcal{E}_1 \equiv \sum_{\kappa \in \mathcal{T}_{h,1}} |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_2 \equiv \sum_{\kappa \in \mathcal{T}_{h,2}} |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_3 \equiv \sum_{\kappa \in \mathcal{T}_{h,3}} |\eta_\kappa^{\text{new}}|$

- Solve local primal and dual problems on elemental patches.
- Boundary data extracted from global primal and dual solutions.



Adaptive Algorithm (h–Adaptivity)

Algorithm 1

Select optimal refinement

$$\max_{i=1,2,3}(|\eta_\kappa^{\text{old}}| - \mathcal{E}_i)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})).$$

Algorithm 2

- Prescribe an h–anisotropy parameter $\theta_h > 1$.
- When

$$\frac{\max_{i=1,2}(\mathcal{E}_i)}{\min_{i=1,2}(\mathcal{E}_i)} > \theta_h,$$

  perform refinement in direction with minimal $\mathcal{E}_i$, $i = 1, 2$.
- else perform isotropic h–refinement.

Adaptive Algorithm (h–Adaptivity)

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NACA0012 Airfoil (Comp. NS)

$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 2^\circ$ and adiabatic wall condition.

Drag coefficients:

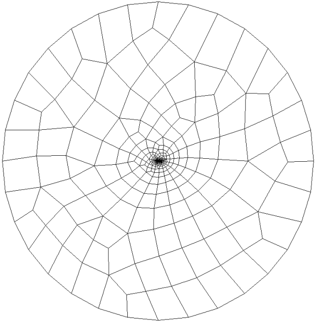
$$J_{c_{\mathrm{dp}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2} \int_S p\,(\mathbf{n} \cdot \psi_d)\mathrm{d}s, \qquad J_{c_{\mathrm{df}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2} \int_S (\boldsymbol{\tau}\,\mathbf{n}) \cdot \psi_d \mathrm{d}s,$$

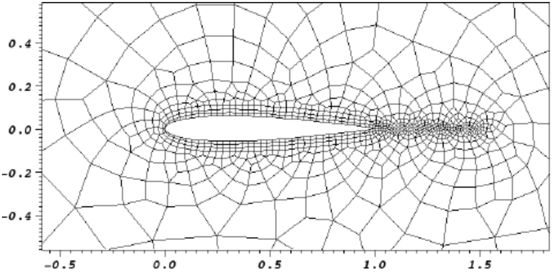
where

$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

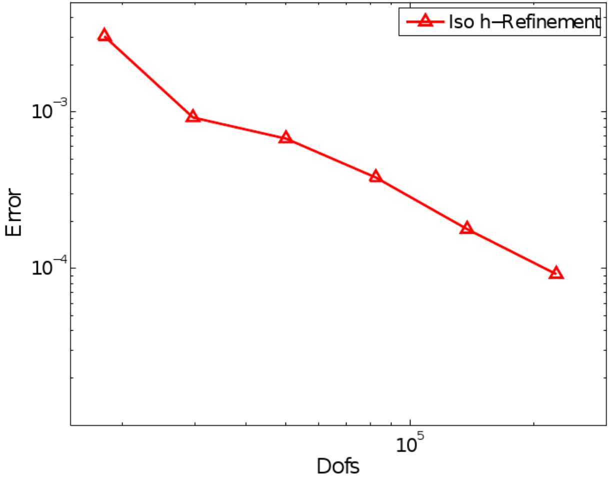
$$\boxed{J_{c_{\mathrm{d}}}(\mathbf{u}) \approx 0.056084.}$$

NACA0012 Airfoil – Unstructured





NACA0012 Airfoil – Unstructured



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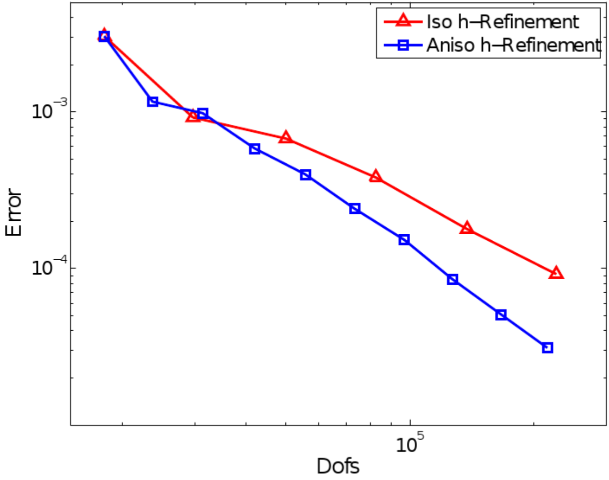
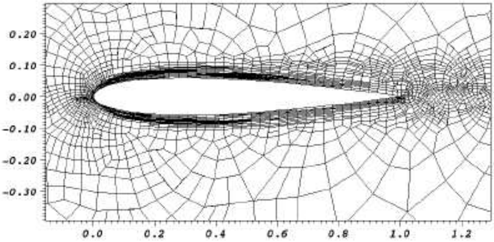
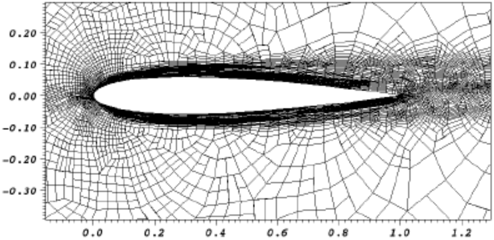


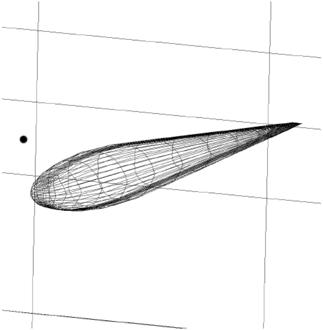


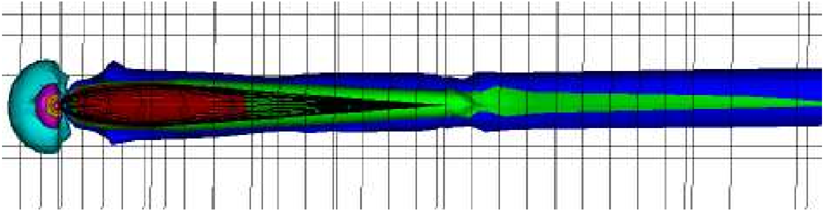
Mesh after 4 adaptive anisotropic refinements, with 3485 elements



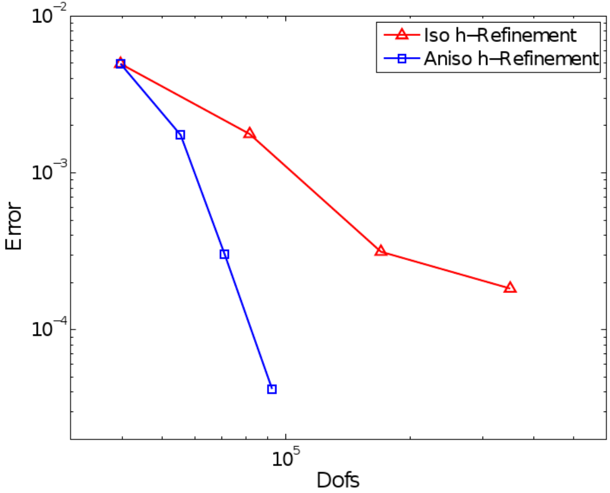
Mesh after 8 adaptive anisotropic refinements, with 10401 elements

- $\mathrm{Ma} = 0.5$.
- $\mathrm{Re} = 5000$.
- $\alpha = 1^\circ$.
- Adiabatic wall condition.
- $J_{C_1}(\mathbf{u}) \approx 0.002565$



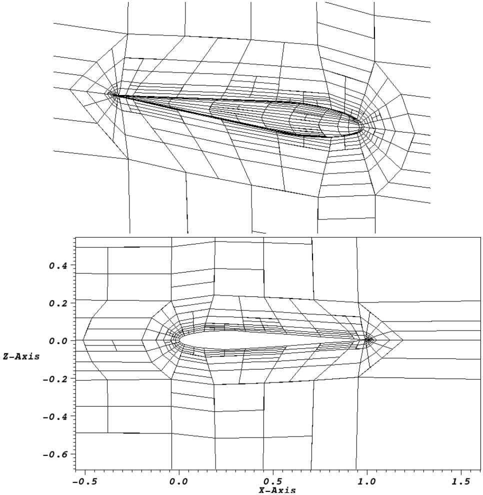


BTC0 (Comp. NS)



Iso h–Refinement
Aniso h–Refinement
10^{-2}
10^{-3}
10^{-4}
Error
10^{5}
Dofs

BTC0 (Comp. NS)

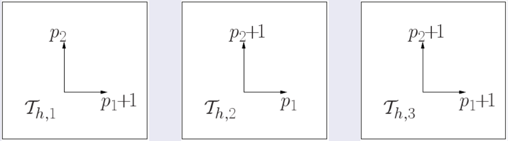


Mesh after 3 adaptive anisotropic refinements, with 2324 elements

Outline

Anisotropic p–Refinement

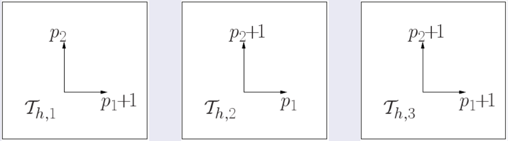
Local Problems



$\mathcal{E}_1 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_2 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_3 \equiv |\eta_\kappa^{\text{new}}|$

Anisotropic p–Refinement

Local Problems



$\mathcal{E}_1 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_2 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_3 \equiv |\eta_\kappa^{\text{new}}|$

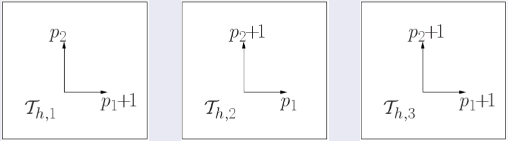
Algorithm 1

Select optimal refinement

$$\max_{i=1,2,3} (|\eta_\kappa^{\text{old}}| - \mathcal{E}_i)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})).$$

Anisotropic p–Refinement

Local Problems



$\mathcal{E}_1 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_2 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_3 \equiv |\eta_\kappa^{\text{new}}|$

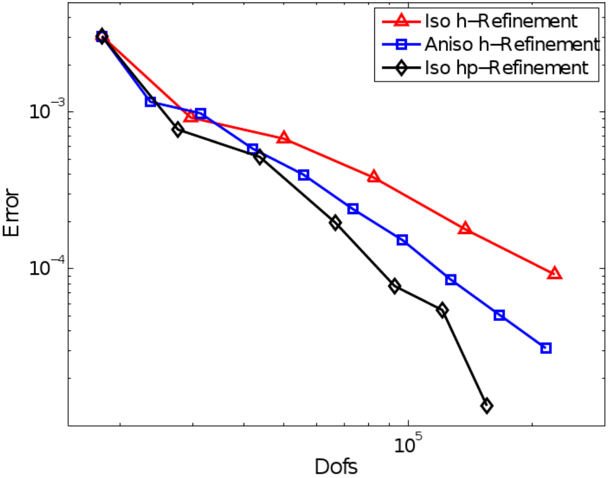
Algorithm 2

- Prescribe a p–anisotropy parameter $\theta_p > 1$
- When

$$\frac{\max_{i=1,2}(\mathcal{E}_i/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})))}{\min_{i=1,2}(\mathcal{E}_i/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})))} > \theta_p,$$

  enrich in polynomial in the direction with minimal $\mathcal{E}_i$, $i = 1, 2$.
- else perform isotropic p–refinement.

NACA0012 Airfoil – Unstructured



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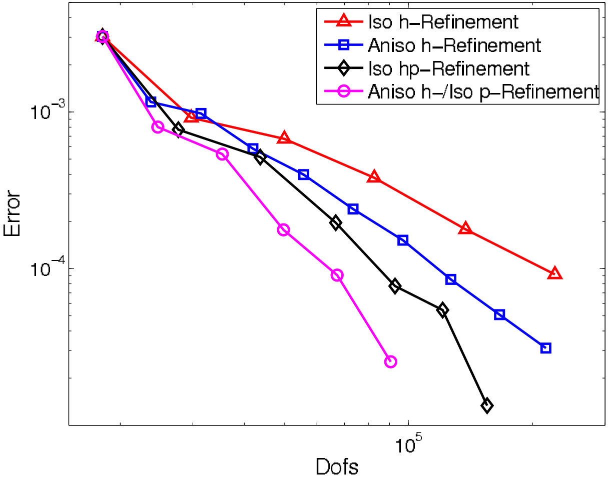

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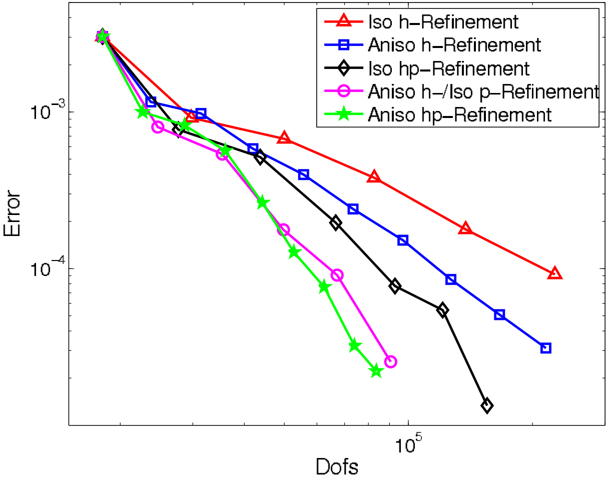

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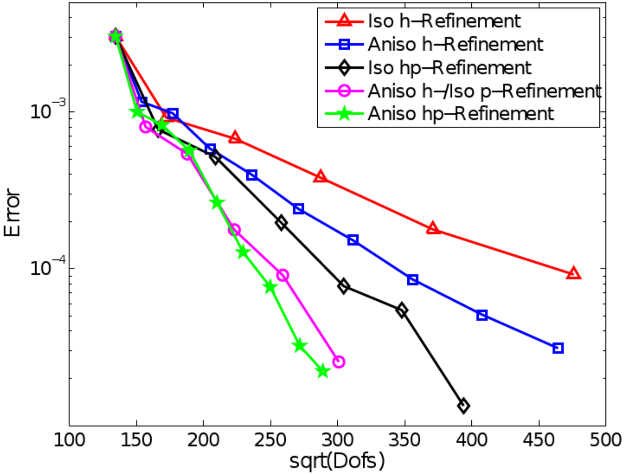
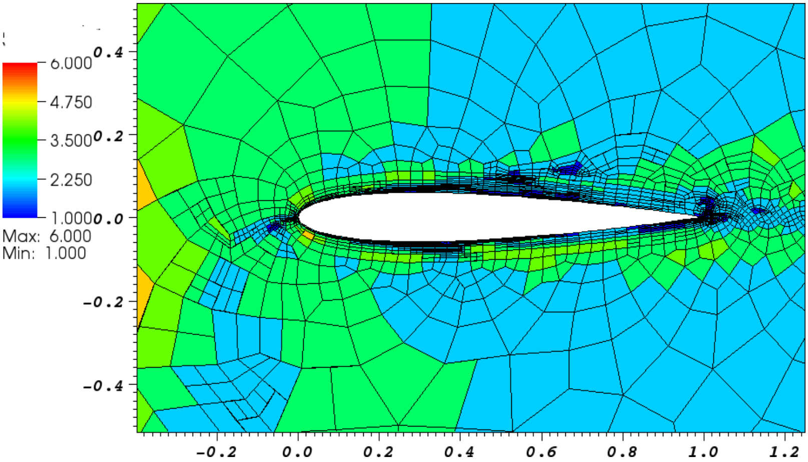
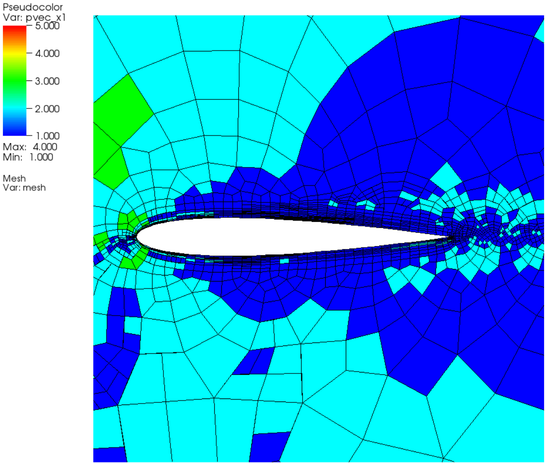


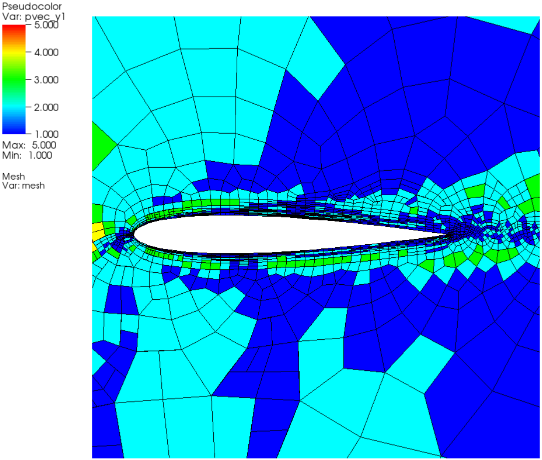


hp–mesh distribution after 6 adaptive (anisotropic *h*–/isotropic *p*–) refinements, with 2835 elements and 118520 degrees of freedom

NACA0012 Airfoil – Unstructured



hp/p_x–mesh distribution after 6 adaptive (anisotropic h–/anisotropic p–) refinements



hp/p_y–mesh distribution after 6 adaptive (anisotropic h–/anisotropic p–) refinements