High-Order/*hp*-Adaptive Multilevel Discontinuous Galerkin Methods

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www.AptoFEM.com

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Outline

1. **Introductions**
   - Aims
2. **DGCFE Method**
   - An overview
   - Finite element space
   - SIP DGCFE method
3. **Adaptivity**
   - Energy Norm Error Estimator
   - Goal–oriented Error Estimator

Outline

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Aims

We want a discontinuous Galerkin (DG) method such that:

1. The number of DOFs is independent of how complicated is the domain
2. A method easy to extend to high-orders.
3. A method such that all adaptive techniques we already use are easy to use with
4. Any new DOF inserted by adaptivity is useful to improve the accuracy and not wasted to describe the domain
5. A preconditioner working on a variety of problems

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DGCFE (Discontinuous Galerkin Composite Finite Element) method:

P. Antonietti, S.G. and P. Houston
hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains
SISC, accepted.

It is an extension of CFE for continuous Galerkin:

W. Hackbusch and S.A. Sauter
Composite finite elements for the approximation of PDEs on domains with complicated micro-structures
Numer. Math., 75, 447–472, 1997.

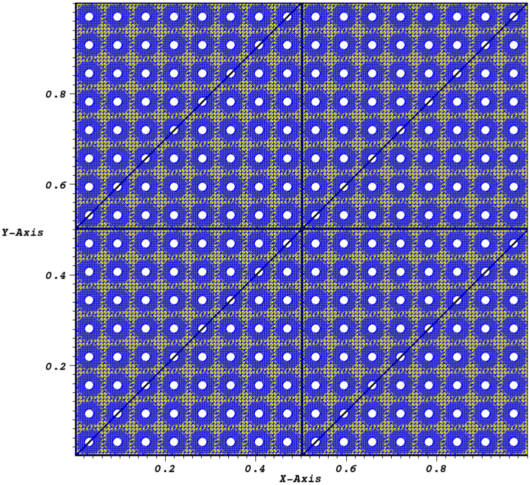
W. Hackbusch and S.A. Sauter
Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results
Comput. Visual Sci., 1, 15–25, 1997.

Two levels

Two meshes:

- Mesh $\mathcal{T}_{h_\ell}$ is a partition of the domain Ω and describes all the details in the domain.
- A coarser mesh $\mathcal{T}_{\mathrm{CFE}}$ which is too coarse to describe the details in the domain Ω.

Two levels



Two levels

Transferring information from the fine level to the coarse level:

1. The geometrical details are not "stored" in the coarse mesh $\mathcal{T}_{\mathrm{CFE}}$ but in the finite element basis functions on the coarse level.
2. Boundary conditions are imposed weakly.
3. The mesh $\mathcal{T}_{\mathrm{CFE}}$ and the correspondent finite element space $V(\mathcal{T}_{\mathrm{CFE}}, p)$ are used to set the size and the sparsity of the linear system.
4. The mesh $\mathcal{T}_{h_\ell}$ and the correspondent finite element space $V(\mathcal{T}_{h_\ell}, p)$ are used to compute the entries of the linear system.

Outline

CFE space

On each CFE element τ_{CFE} we set up a polynomial space of order p.
The domain of each element τ_{CFE} is the union of its children.
Also on the children elements of τ_{CFE} we set up polynomial spaces of same order p. In this way the basis functions of τ_{CFE} can be expressed using the basis functions of the children:

$$\phi_{\mathtt{CFE},i} := \sum_{j=1,\ldots,\dim(V(\mathcal{T}_{h_\ell},p))} \alpha_{i,j}\, \phi_{h_\ell,j} \ .$$

Outline

Durham University

Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain, and let $\mathcal{T}_{\mathrm{CFE}} = \{\kappa\}$ be a subdivision of Ω. Let $u_h \in V(\mathcal{T}_{\mathrm{CFE}}, p)$ be the composite discontinuous Galerkin approximation to u and suppose that $u|_\kappa \in H^{k_\kappa}(\kappa)$ for each $\kappa \in \mathcal{T}_{\mathrm{CFE}}$ for integers $k_\kappa \geq 1$. Then, the following error bound holds

$$|||\, u - u_h \,|||_{\mathrm{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \frac{h_\kappa^{2s_\kappa}}{h_F^2} \frac{1}{p_\kappa^{2k_\kappa - 3}} \|\mathfrak{E}\tilde{u}\|_{H^{k_\kappa}(\hat{\kappa})}^2,$$

for any integers s_κ, $1 \leq s_\kappa \leq \min(p_\kappa + 1, k_\kappa)$, and $p_\kappa \geq 1$. Here, C is a positive constant that depends only on the dimension d.

$$|||\, v \,|||_{\mathrm{DG}}^2 = \sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \|\nabla v\|_{L_2(\kappa)}^2 + \sum_{F \in \mathcal{F}_{\mathrm{CFE}}^{\mathcal{I}} \cup \mathcal{F}_{\mathrm{CFE}}^{\mathcal{B}}} \|\sigma^{1/2} [\![v]\!]\|_{L_2(F)}^2.$$

Problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned}$$

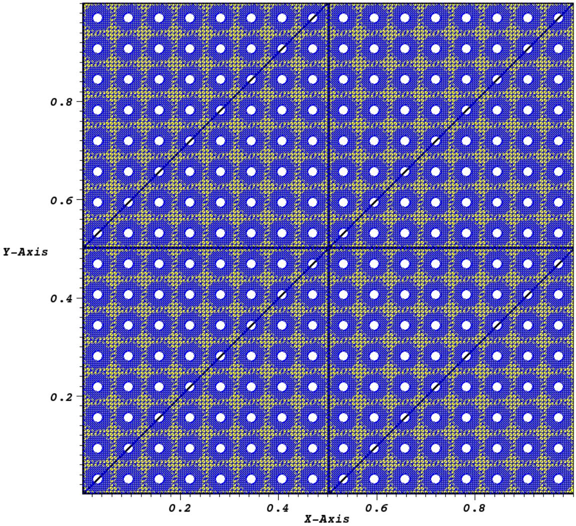
Solution:

$$u = \sin(\pi x)\cos(\pi y).$$

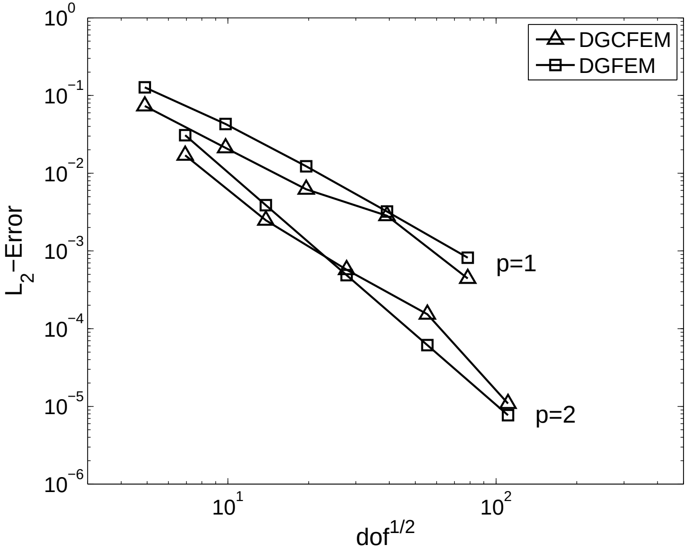
Ω has micro-structures.

P. Antonietti, S.G. and P. Houston
hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains
SISC, accepted.

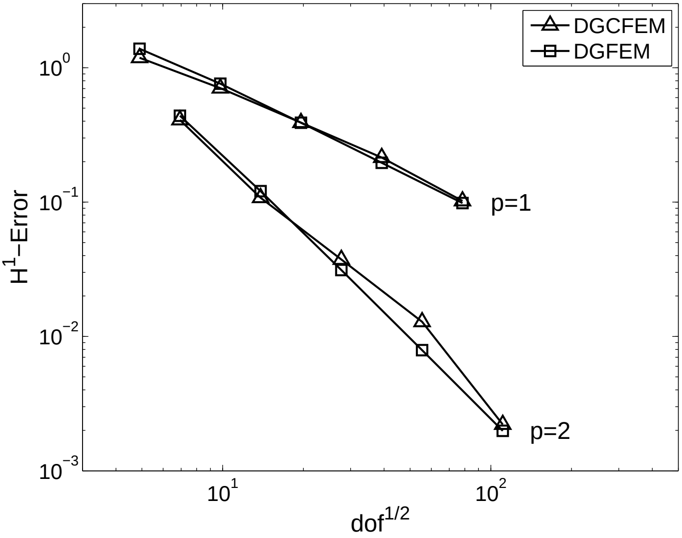
Domain



Convergence in L^2



Convergence in H^1



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Pros & Cons

Cons:

1. **Not Flexible:** only the norm of the error can be predicted
2. **No Sharp Bound:** there is a hidden constant

Pros:

1. **Cheap:** easy to compute

$$|||\, u - u_h \,|||_{\mathtt{DG}} \leq C \left(\sum_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} (\eta_\kappa^2 + \mathcal{O}_\kappa^2) \right)^{\frac{1}{2}}, \qquad (1)$$

where the local error indicators η_κ, $\kappa \in \mathcal{T}_{\mathtt{CFE}}$, are defined by

$$\begin{aligned} \eta_\kappa^2 &= h_\kappa^2 p_\kappa^{-2} \|\Pi f + \Delta u_h\|_{L_2(\kappa)}^2 \\ &\quad + \sum_{F \subset \partial\kappa \setminus \partial\Omega} h_\kappa^2 h_F^{-1} p_\kappa^{-1} \|[\![\nabla u_h]\!]\|_{L_2(F)}^2 + \sigma h_\kappa^2 h_F^{-2} p_\kappa \|[\![u_h]\!]\|_{L_2(\partial\kappa)}^2, \end{aligned} \qquad (2)$$

and the data oscillation term $\mathcal{O}_\kappa$ is given by

$$\mathcal{O}_\kappa = h_\kappa^2 p_\kappa^{-2} \|f - \Pi f\|_{L_2(\kappa)}^2.$$

S.G. and P. Houston
hp-Adaptive Composite Discontinuous Galerkin Methods for Elliptic Problems on Complicated Domains
Numerical Methods for Partial Differential Equations, Accepted.

L–Shape

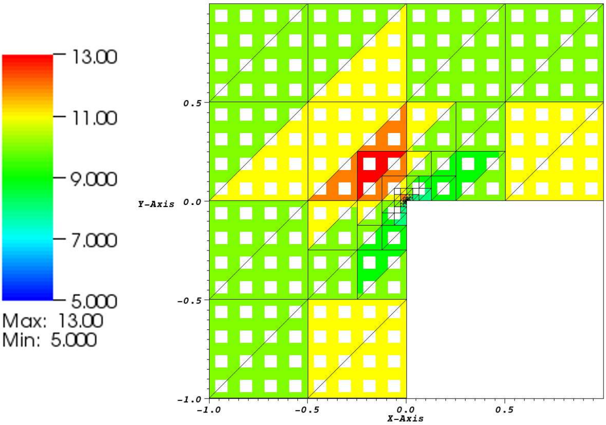

Figure: 16th adapted mesh to approximate the solution $u = r^{2/3}\sin(2\varphi/3)$.

L–Shape

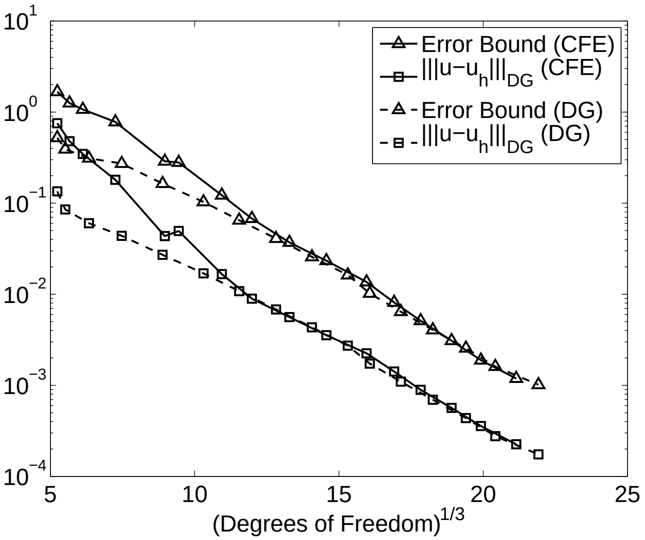

Figure: Convergence

L–Shape

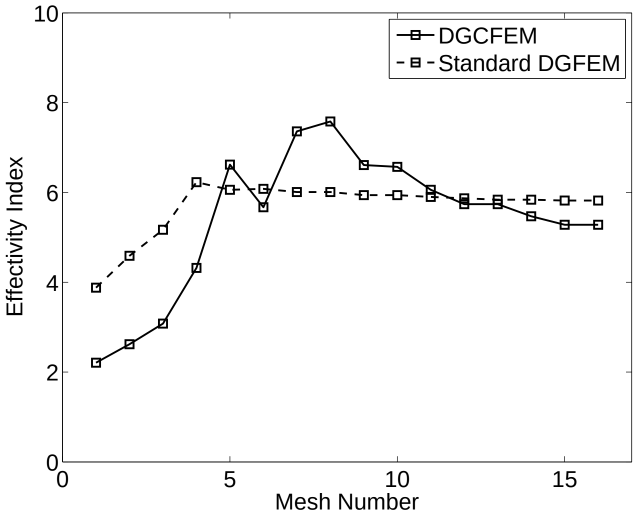

Figure: Effectivity

Plateau example

The rhs is defined as $f = 0$ in the subdomain $[0.25, 0.75]^2$ and $f = 1$ everywhere else.

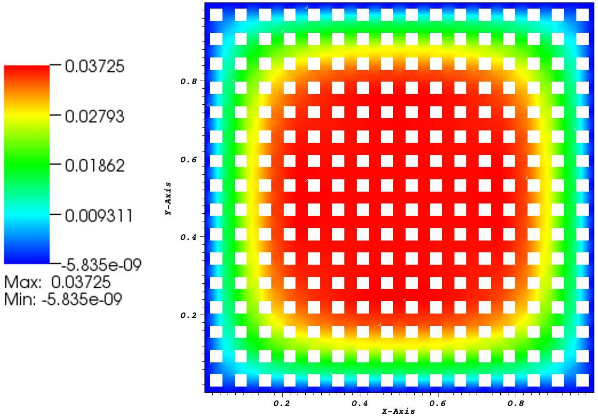

Figure: Solution.

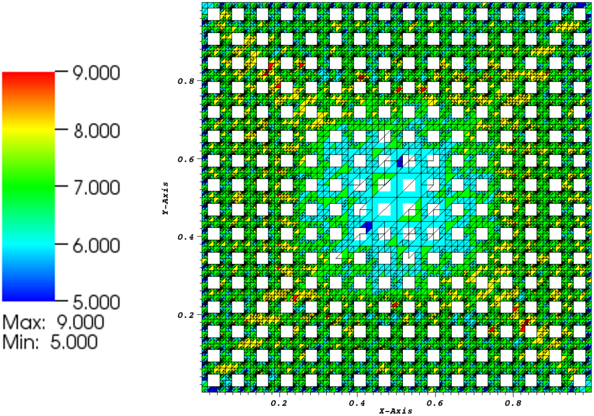

Figure: 18th adapted mesh.

Plateau example

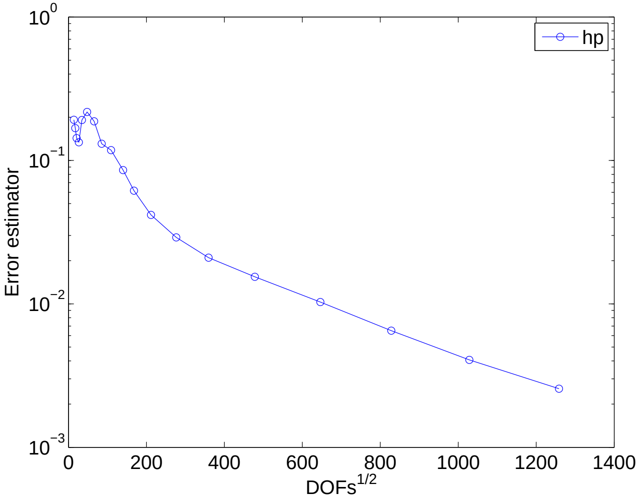

Figure: Convergence

Outline

Cons:

1. **Expensive:** a dual problem should be solved

Pros:

1. **Flexible:** the quantity of interest can be chosen
2. **Sharp Bound:** the upper bound has constant 1

$$|J(u) - J(u_{hp})| \lesssim \sum_{\kappa \in \mathcal{T}_h} |\tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp})| \ .$$

3. **Accurate:** it is possible to compute an accurate estimation of the error in the quantity of interest

$$J(u) - J(u_{hp}) \approx \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp}) \ ,$$

4. **Improved Accuracy:** it is possible to improve the accuracy of the quantity of interest

$$J(u) \approx J(u_{hp}) + \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp}) \ ,$$

Navier–Stokes equations

$$
\begin{aligned}
-\frac{1}{Re}\nabla^2\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p &= \mathbf{0}, \quad \text{in } \Omega, && (3)\\
\nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, && (4)
\end{aligned}
$$

with boundary conditions

$$
\mathbf{u} = \mathbf{g}_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \frac{1}{Re}\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \partial\Omega_{\mathrm{N}}, \qquad (5)
$$

S.G. and P. Houston
Goal–Oriented Adaptive Composite Discontinuous Galerkin Methods for Incompressible Flows
Journal of Computational and Applied Mathematic (2014) 270, 32-42.

T-pipe

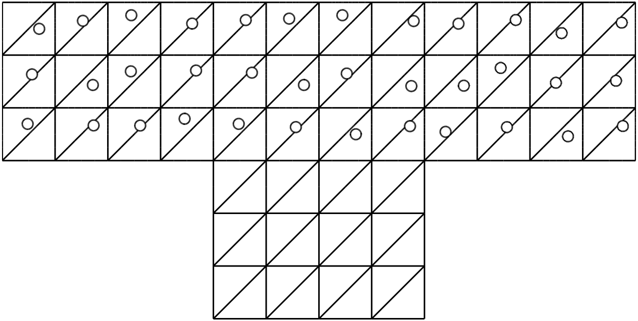

Figure: First Mesh

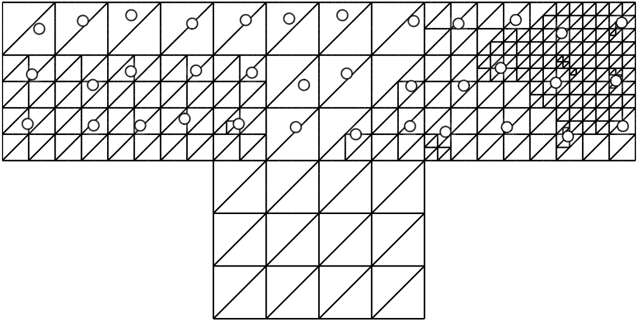

Figure: Fourth Mesh

T-pipe

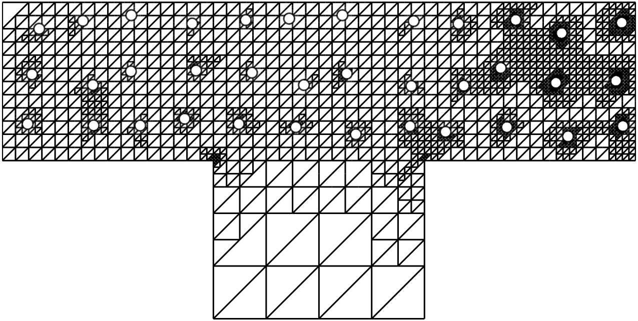

Figure: Eighth Mesh

T-pipe

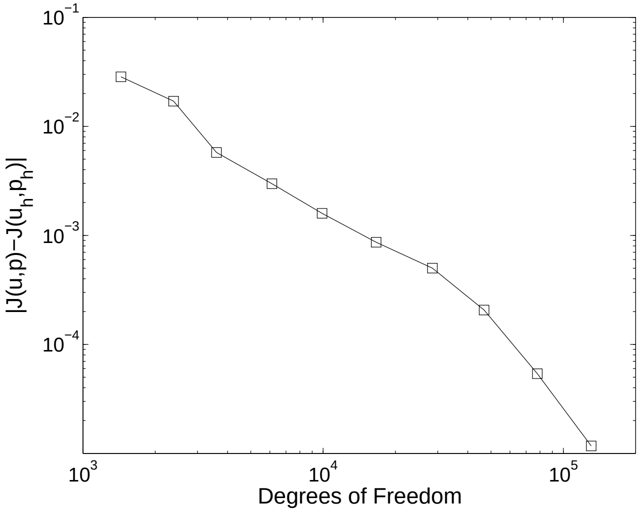

Figure: Convergence

T-pipe

No of Eles	No of Dofs	$J(\mathbf{u},p) - J(\mathbf{u}_h,p_h)$	$\sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \eta_\kappa$	θ
96	1440	-2.849E-2	-2.534E-2	0.89
159	2385	-1.702E-2	-1.430E-2	0.84
240	3600	-5.755E-3	-3.419E-3	0.59
408	6120	-2.974E-3	-1.554E-3	0.52
660	9900	-1.592E-3	-7.969E-4	0.50
1108	16620	-8.644E-4	-3.853E-4	0.45
1901	28515	-5.008E-4	-2.842E-4	0.57
3118	46770	-2.068E-4	-1.468E-4	0.71
5196	77940	-5.390E-5	-4.716E-5	0.87
8708	130620	-1.172E-5	-1.172E-5	1.00

Table: Adaptive algorithm.