High-Order/*hp*-Adaptive Multilevel Discontinuous Galerkin Methods with applications in fluid dynamics

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Joint work with
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www.AptoFEM.com

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Outline

1. **Introductions**
   - An overview
2. **Adaptivity**
   - Goal–oriented Error Estimator
3. **CFEDG Preconditioner**
   - Additive Schwarz Preconditioner
   - Compressible Navier-Stokes

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DGCFE (Discontinuous Galerkin Composite Finite Element) method:

- P. Antonietti, S.G. and P. Houston
  hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains
  SISC, 35(3), A1417-A1439.

It is an extension of CFE for continuous Galerkin:

- W. Hackbusch and S.A. Sauter
  Composite finite elements for the approximation of PDEs on domains with complicated micro-structures
  Numer. Math., 75, 447–472, 1997.

- W. Hackbusch and S.A. Sauter
  Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results
  Comput. Visual Sci., 1, 15–25, 1997.

Two levels

Two meshes:

- Mesh $\mathcal{T}_{h_\ell}$ is a partition of the domain Ω and describes all the details in the domain.
- A coarser mesh $\mathcal{T}_{\mathrm{CFE}}$ which is too coarse to describe the details in the domain Ω.

Two levels

Transferring information from the fine level to the coarse level:

1. The geometrical details are not "stored" in the coarse mesh $\mathcal{T}_{\mathrm{CFE}}$ but in the finite element basis functions on the coarse level.
2. Boundary conditions are imposed weakly.
3. The mesh $\mathcal{T}_{\mathrm{CFE}}$ and the correspondent finite element space $V(\mathcal{T}_{\mathrm{CFE}}, p)$ are used to set the size and the sparsity of the linear system.
4. The mesh $\mathcal{T}_{h_\ell}$ and the correspondent finite element space $V(\mathcal{T}_{h_\ell}, p)$ are used to compute the entries of the linear system.

T-pipe

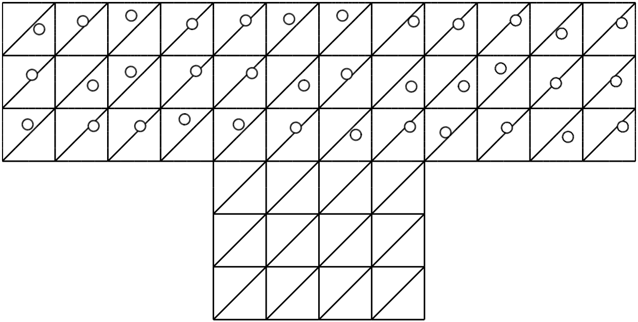

Figure : Coarse Level $\mathcal{T}_{h_\ell}$ Mesh

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Pros & Cons

Cons:

1. **Expensive:** a dual problem should be solved

Pros:

1. **Flexible:** the quantity of interest can be chosen
2. **Sharp Bound:** the upper bound has constant 1

$$|J(u) - J(u_{hp})| \lesssim \sum_{\kappa \in \mathcal{T}_h} |\tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp})| \ .$$

3. **Accurate:** it is possible to compute an accurate estimation of the error in the quantity of interest

$$J(u) - J(u_{hp}) \approx \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp}) \ ,$$

4. **Improved Accuracy:** it is possible to improve the accuracy of the quantity of interest

$$J(u) \approx J(u_{hp}) + \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp}) \ ,$$

Navier–Stokes equations

$$
\begin{aligned}
-\frac{1}{Re}\nabla^2\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p &= \mathbf{0}, \quad \text{in } \Omega, \qquad (1)\\
\nabla\cdot\mathbf{u} &= 0, \quad \text{in } \Omega, \qquad (2)
\end{aligned}
$$

with boundary conditions

$$
\mathbf{u} = \mathbf{g}_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \frac{1}{Re}\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \partial\Omega_{\mathrm{N}}, \qquad (3)
$$

S.G. and P. Houston
Goal–Oriented Adaptive Composite Discontinuous Galerkin Methods for Incompressible Flows
Journal of Computational and Applied Mathematic (2014) 270, 32-42.

T-pipe

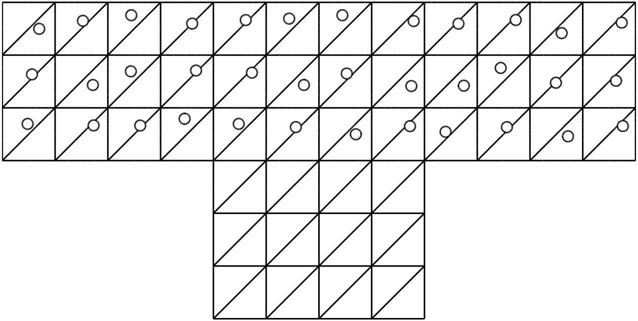

Figure : First Mesh

T-pipe

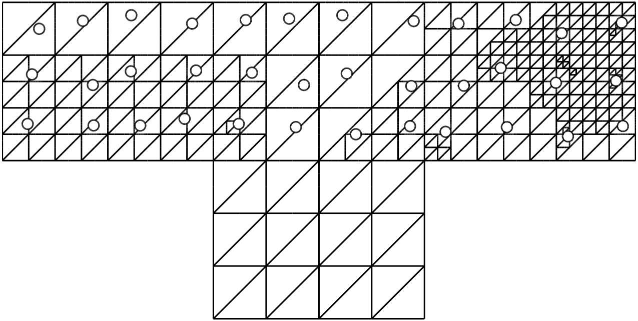

Figure : Fourth Mesh

T-pipe

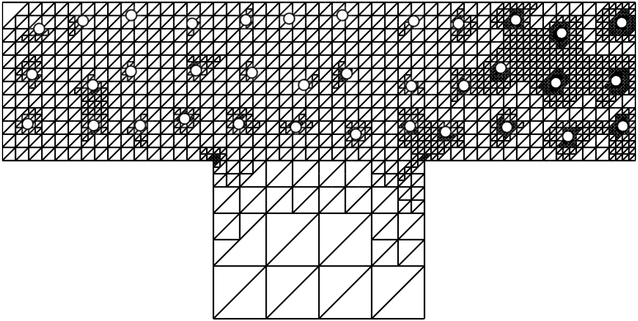

Figure : Eighth Mesh

T-pipe

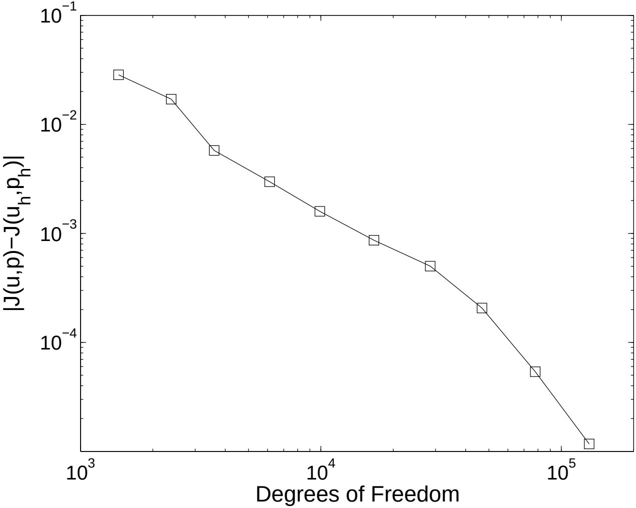

Figure : Convergence

T-pipe

No of Eles	No of Dofs	$J(\mathbf{u}, p) - J(\mathbf{u}_h, p_h)$	$\sum_{\kappa \in \mathcal{T}_{\mathrm{CFE}}} \eta_\kappa$	θ
96	1440	-2.849E-2	-2.534E-2	0.89
159	2385	-1.702E-2	-1.430E-2	0.84
240	3600	-5.755E-3	-3.419E-3	0.59
408	6120	-2.974E-3	-1.554E-3	0.52
660	9900	-1.592E-3	-7.969E-4	0.50
1108	16620	-8.644E-4	-3.853E-4	0.45
1901	28515	-5.008E-4	-2.842E-4	0.57
3118	46770	-2.068E-4	-1.468E-4	0.71
5196	77940	-5.390E-5	-4.716E-5	0.87
8708	130620	-1.172E-5	-1.172E-5	1.00

Table : Adaptive algorithm.

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Additive Schwarz Preconditioner

- $\mathcal{T}_S = \{\Omega_i\}_{i=1}^N$ a family of partitions of Ω into N non-overlapping domains, such that $\overline{\Omega} = \cup_{i=1}^N \overline{\Omega}_i$.
- Mesh hierarchy : $\mathcal{T}_S \subseteq \mathcal{T}_{\mathrm{CFE}} \subseteq \mathcal{T}_h$.
- Additive Schwarz Preconditioner: $P_{\mathrm{ad}} := \sum_{i=0}^N P_i$.
- P_0 is the coarse level problem computed on $\mathcal{T}_{\mathrm{CFE}}$.

P. Antonietti, S. G. and P. Houston
Domain Decomposition Preconditioners for Discontinuous Galerkin Methods for Elliptic Problems on Complicated Domains
Journal of Scientific Computing 60(1), 203-227.

Outline

- Subsonic viscous flow around a NACA0012 airfoil.
- Mach 0.5.
- angle of attack $\alpha = 2^\circ$.
- Reynolds number $\text{Re} = 5000$.
- Newton–GMRES tolerance 10^{-8}.
- absolute tolerance for GMRES 10^{-8}.
- restart every 500 iterations.

Compressible Navier-Stokes

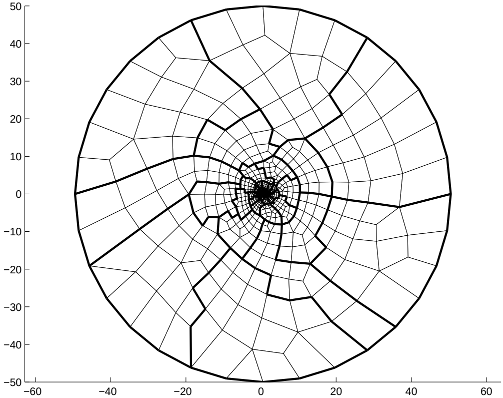

Figure : Mesh 5 partitioned into 500 regions using METIS.

Compressible Navier-Stokes

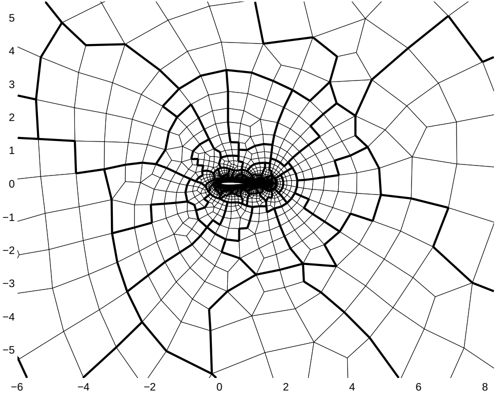

Figure : Mesh 5 partitioned into 500 regions using METIS.

Compressible Navier-Stokes

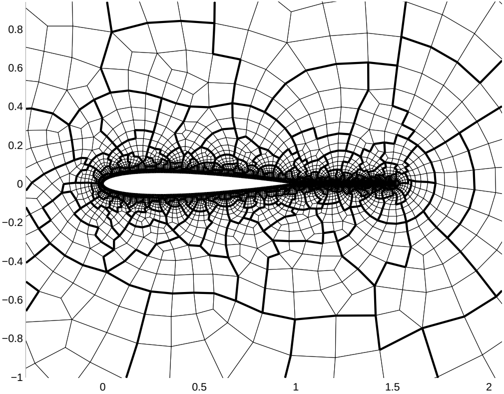

Figure : Mesh 5 partitioned into 500 regions using METIS.

	500	1000	2000	4000	8000
Mesh 1	124(936,10)	-	-	-	-
Mesh 2	186(1303,9)	121(800,9)	-	-	-
Mesh 3	310(1957,9)	168(1150,9)	116(700,9)	-	-
Mesh 4	519(3136,9)	278(1796,9)	151(1034,9)	95(646,9)	-
Mesh 5	933(5604,9)	492(3034,9)	276(1785,9)	162(1090,9)	103(687,9)

Table : GMRES iteration counts for the preconditioner, on unstructured meshes using coarse meshes created with METIS and a subdomain partition generated with METIS consisting of $N = 250$ domains.

	GMRES+ILU
Mesh 1	130 (927,9)
Mesh 2	217 (1410,9)
Mesh 3	431 (2956,9)
Mesh 4	1086 (5938,9)
Mesh 5	* (*,*)

Table : GMRES+ILU iteration counts.

Durham University

$$|||\, u - u_h \,|||_{\mathrm{DG}} \leq C \left(\sum_{\kappa \in \mathcal{T}_{\mathtt{CFE}}} (\eta_\kappa^2 + \mathcal{O}_\kappa^2) \right)^{\frac{1}{2}}, \tag{4}$$

where the local error indicators η_κ, $\kappa \in \mathcal{T}_{\mathtt{CFE}}$, are defined by

$$\begin{aligned} \eta_\kappa^2 &= h_\kappa^2 p_\kappa^{-2} \|\Pi f + \Delta u_h\|_{L_2(\kappa)}^2 \\ &+ \sum_{F \subset \partial\kappa \setminus \partial\Omega} h_\kappa^2 h_F^{-1} p_\kappa^{-1} \|[\![\nabla u_h]\!]\|_{L_2(F)}^2 + \sigma h_\kappa^2 h_F^{-2} p_\kappa \|[\![u_h]\!]\|_{L_2(\partial\kappa)}^2, \end{aligned} \tag{5}$$

and the data oscillation term $\mathcal{O}_\kappa$ is given by

$$\mathcal{O}_\kappa = h_\kappa^2 p_\kappa^{-2} \|f - \Pi f\|_{L_2(\kappa)}^2.$$

S.G. and P. Houston
hp-Adaptive Composite Discontinuous Galerkin Methods for Elliptic Problems on Complicated Domains
Numerical Methods for Partial Differential Equations, 30(4), 1342-1367.

L–Shape

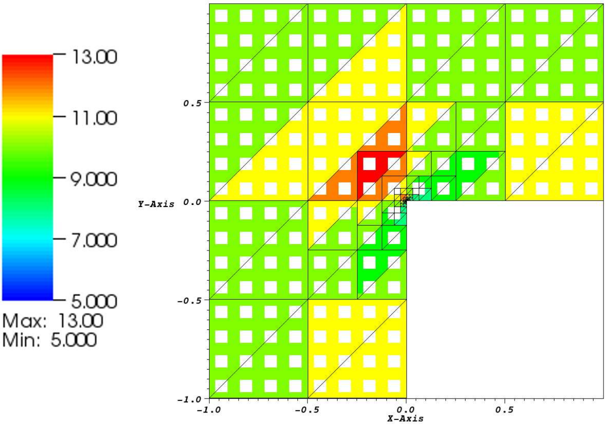

Figure : 16th adapted mesh to approximate the solution $u = r^{2/3}\sin(2\varphi/3)$.

L–Shape

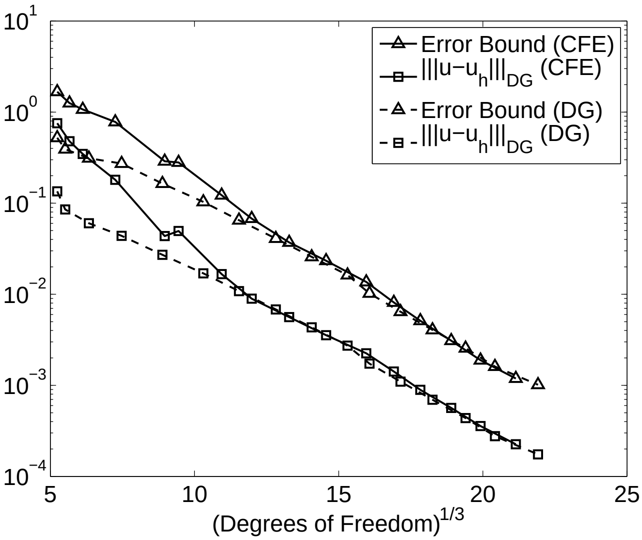

Figure : Convergence

L–Shape

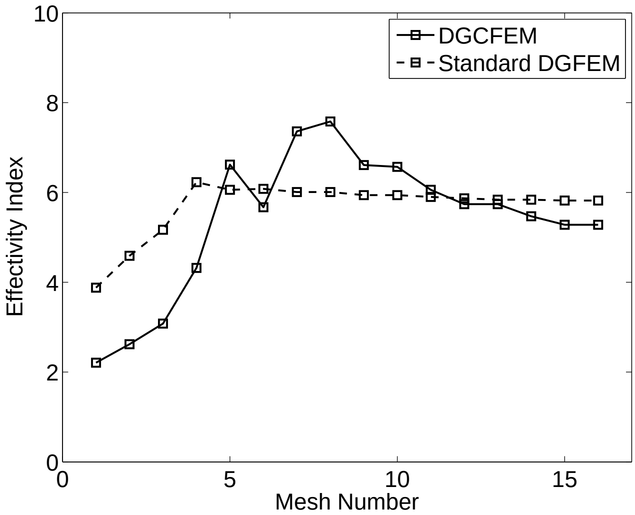

Figure : Effectivity