Adaptive methods for photonic crystal fibers

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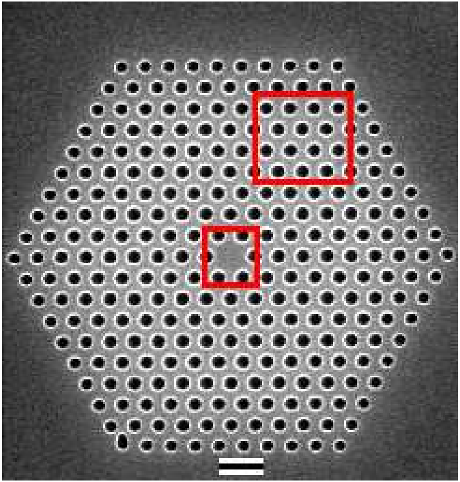
What is a Photonic Crystal Fiber?



It can be possible for light of certain frequencies to be expelled for all directions - light of these frequencies simply will not propagate in the material. This situation is called a "photonic band gap".
A periodic structure could be used to keep light of certain frequencies in the center of the fiber.

Maxwell's Equations

$$\begin{cases} \nabla \times \mathbf{E} = -\frac{i\omega}{c}\mu\mathbf{H}, \\ \nabla \times \mathbf{H} = \frac{i\omega}{c}\varepsilon\mathbf{E}, \\ \nabla \cdot \mu\mathbf{H} = 0, \\ \nabla \cdot \varepsilon\mathbf{E} = 0. \end{cases}$$

$$\mu = 1,$$

$$\varepsilon = \begin{pmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ \epsilon_{21} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{pmatrix},$$

$$\epsilon_{12} = \epsilon_{21}$$

TM Case:

$\mathbf{E} = (0, 0, u(x, y))$

$\mathbf{E} = (u_y(x, y), -u_x(x, y), 0)$

TE Case:

$\mathbf{H} = (0, 0, u(x, y))$

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TE case

$$\mathbf{H} = (0, 0, u(x, y)),$$

$$\nabla \times (\varepsilon^{-1} \nabla \times \mathbf{H}) + \frac{\omega^2}{c^2}\mathbf{H} = 0.$$

$$\nabla \cdot (M \nabla u) + \lambda u = 0,$$

$$M = \frac{1}{\epsilon_{11}\epsilon_{22} - \epsilon_{12}^2} \begin{pmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{pmatrix}.$$

Floquet transform

$$g \in L^2(\mathbb{R}^2) \quad (\mathcal{F}g)(\alpha, \mathbf{x}) = e^{-i\alpha\cdot\mathbf{x}} \sum_{\mathbf{n}\in\mathbb{Z}^2} g(\mathbf{x}-\mathbf{n}) e^{i\alpha\cdot n}$$

$$\alpha \in [-\pi, \pi]^2, x \in [0,1]^2$$

$\Rightarrow$Now the domain of x is a torus and α is a parameter.

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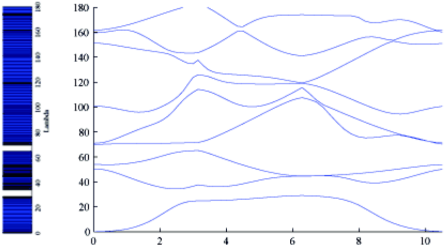
Lemma: $\mathcal{F}(\nabla g) = (\nabla + i\alpha)\mathcal{F}g.$

TE: $\nabla \cdot (M\nabla u) + \lambda u = 0, \quad x \in \mathbb{T}, \alpha \in [-\pi, \pi]^2.$



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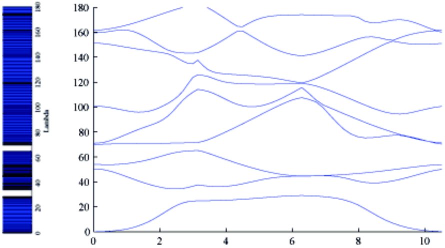

Why a Posteriori Error Estimator for TE case?

Reasons:

- Optimize Computational Power
- Regularity in each Subdomain is $H^{1+1/2+\beta}(\mathbb{T})$, $\beta > 0$
- Solutions could oscillate a lot.

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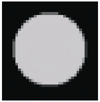
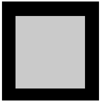
Posteriori Error Estimator

$$a_\alpha(u,v) = \lambda(u,v), \qquad \text{(Continuous)}$$

$$a_\alpha(u_h,v_h) = \lambda_h(u_h,v_h), \qquad \text{(Discrete)}$$





- $\mathcal{T}_h$ (Mesh resolves the interface), $V_h \subset H^1(\mathbb{T},\mathbb{C})$ (Finite Element Space)
- $\eta(\lambda_h,u_h) = \left(\sum_{\tau\in\mathfrak{T}_h} \eta_\tau^2(\lambda_h,u_h)\right)^{1/2}$ (Error Estimator)

Properties:

1. $\|e\|_1 \le C\eta + h.o.t.$ (Reliability)
2. $\lambda_h - \lambda \le C\eta^2 + h.o.t.$ (Reliability)
3. $\eta \le C\|e_h\|_1 + h.o.t.$ (Efficiency)

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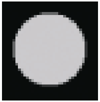

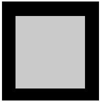

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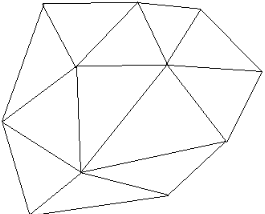
Residuals

(with isotropic dielectricity ε)

$$\eta = \left(\sum_{\tau \in \mathfrak{T}_h} \eta_\tau^2(\lambda_h, u_h) \right)^{1/2},$$

$$R_I(u_h, \lambda_h)(x) := ((\nabla + i\alpha) \cdot \varepsilon^{-1}(\nabla + i\alpha)u_h + \lambda_h u_h)(x),$$

$$\begin{aligned} R_E(u_h)(x) \quad &:= ([\hat{n} \cdot \varepsilon^{-1} \nabla u_h])(x) \\ &+([\hat{n} \cdot \varepsilon^{-1} i\alpha u_h])(x). \end{aligned}$$



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$$\eta = \left(\sum_{\tau \in \mathfrak{T}_h} {h_\tau}^2 \varepsilon_\tau \| R_I(u_h, \lambda_h) \|^2_{0,\tau} + \sum_{e \in \mathfrak{F}_h} h_e \varepsilon_e \| R_E(u_h) \|^2_{0,e} \right)^{1/2}.$$

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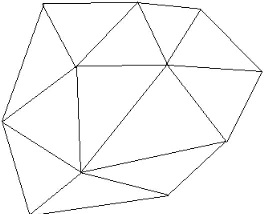

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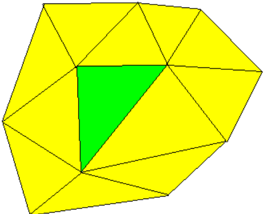

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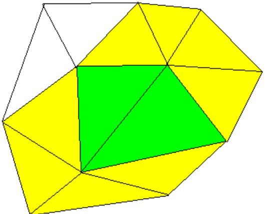

Shift

Proposition 1:(Gårding's inequality)

For any constant $K \geq \varepsilon_{max}\left(\frac{1}{2}+|\alpha|^2\right)$, we have

$$a_\alpha(u,u)+K\|u\|^2_{0,\mathbb{T}} \geq \frac{1}{2}\|u\|^2_{1,\varepsilon,\mathbb{T}} \quad \forall u \in H^1(\mathbb{T},\mathbb{C}).$$

Define:

$$\mu \equiv \lambda - K$$

$$\mu_h \equiv \lambda_h - K$$

Non-linearity

LEMMA 2:

$$(\mu u - \mu_h u_h, e)_{0,\mathbb{T}} = \frac{\lambda + \lambda_h + 2K}{2}(e,e)_{0,\mathbb{T}} + i(\lambda_h - \lambda) Im(u, u_h)_{0,\mathbb{T}}.$$

LEMMA 3:

$$a_\alpha(e,v) + K(e,v)_{0,\mathbb{T}} - (\lambda u - \lambda_h u_h, v)_{0,\mathbb{T}} = \sum_{\tau \in \mathfrak{T}_h} \int_\tau R_I(u_h, \lambda_h)\bar{v}\, d\mathbf{t} - \sum_{e \in \mathfrak{F}_h} \int_e R_E(u_h)\bar{v}\, d\gamma.$$

See [6].

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Reliability

THEOREM 4:

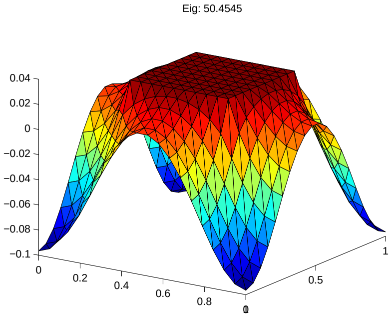
$$\sqrt{a_\alpha(e,e)} \le C_{r1}\eta + \frac{\lambda + \lambda_h + 2K}{2} \frac{(e,e)_{0,\mathbb{T}}}{\sqrt{a_\alpha(e,e)}}.$$

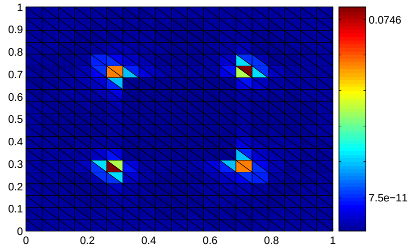
THEOREM 5:

$$\lambda_h - \lambda \quad \le C_{r1}^2\eta^2 + C_{r1}\eta \frac{\lambda_h + \lambda + 2K}{2} \frac{(e,e)_{0,\mathbb{T}}}{\sqrt{a_\alpha(e,e)}}$$

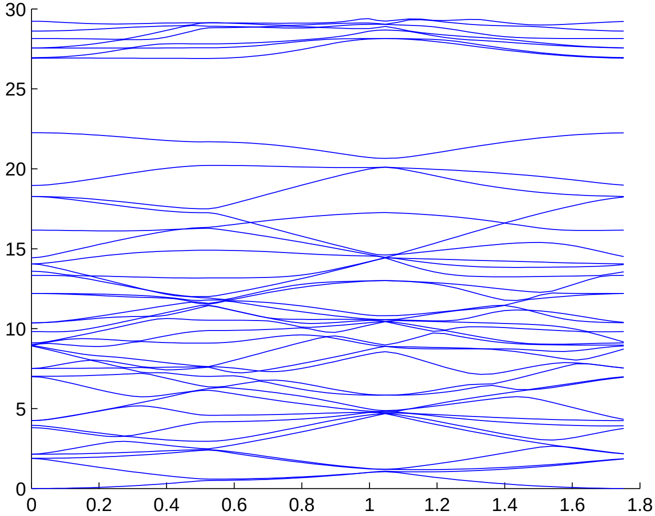
$$+ \frac{\lambda_h - \lambda}{2}(e,e)_{0,\mathbb{T}}.$$

Numerical Results

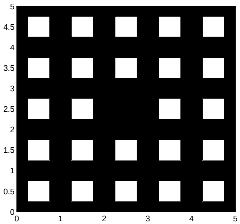


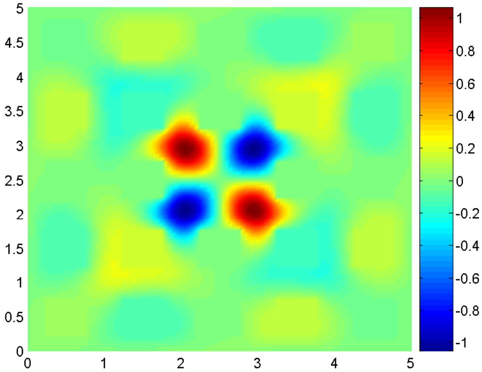


Supercell with defect

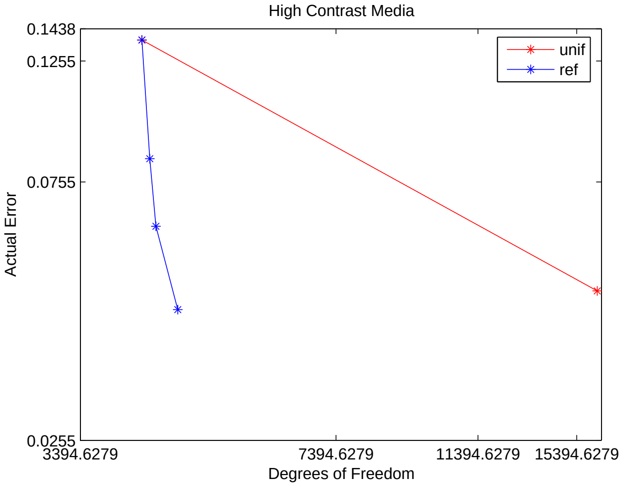


Trapped mode





Convergence



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