Convergence of adaptive finite element methods for elliptic eigenvalue problems with applications in Photonic Crystals

Stefano Giani and Ivan G. Graham

School of Mathematical Sciences
University of Nottingham

MFO, August 2009

Introduction
- Model Problem

Finite Element Methods (FEMs)
- Finite Element Methods (FEMs)

A posteriori error estimator
- Reliability
- Efficiency
- Adaptivity

Convergence Proof
- The Convergent Method
- Convergence Proof

Numerics
- Photonic Crystal Fibers
- Periodic Structure
- Defect modes

Summary

Model Problem

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ (or a bounded polyhedral domain in $\mathbb{R}^3$)

Problem: seek eigenpairs (λ, u) of the problem

$$\begin{cases} -\nabla \cdot (\mathcal{A}\, \nabla u) = \lambda\, \mathcal{B}\, u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that $\mathcal{A}$ and $\mathcal{B}$ are both piecewise constant on Ω and that:

$$\forall \xi \in \mathbb{R}^d \quad \text{with} \quad |\xi| = 1, \quad \forall x \in \Omega, \quad 0 < \underline{a} \leq \xi^T \mathcal{A}(x) \xi \leq \overline{a}\,,$$

and $\mathcal{A}$ is symmetric and

$$\forall x \in \Omega, \quad 0 < \underline{b} \leq \mathcal{B}(x) \leq \overline{b}\,.$$

Variational Formulation

$$\int_\Omega \mathcal{A}\,\nabla u \cdot \nabla v\,dx \;=\; \lambda \int_\Omega \mathcal{B}\,u\,v\,dx\,.$$

$$\begin{aligned} a(u,v) &:= \int_\Omega \mathcal{A}\,\nabla u \cdot \nabla v\,dx\,,\\ |||v|||_\Omega &:= a(v,v)^{1/2}\,,\\ b(u,v) &:= \int_\Omega \mathcal{B}\,u\,v\,dx\,. \end{aligned}$$

Variational Problem: seek eigenpairs $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that

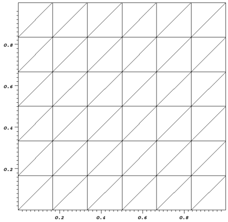
$$\begin{cases} a(u,v) = \lambda b(u,v) \quad \text{for all } v \in H_0^1(\Omega)\,,\\ \|u\|_{0,\Omega} = 1\,. \end{cases}$$

The Ritz-Galerkin Method

Let V_n be a finite dimensional space such that: $V_n \subset H_0^1(\Omega)$:
Seek eigenpairs $(\lambda_n, u_n) \in \mathbb{R} \times V_n$ such that

$$\begin{cases} a(u_n, v_n) = \lambda_n b(u_n, v_n) & \text{for all } v_n \in V_n \ , \\ \|u_n\|_{0,\Omega} = 1 \ . \end{cases}$$

- $\mathcal{T}_n$ conforming and shape regular triangulation of Ω,
- V_n space of piecewise linear functions over $\mathcal{T}_n$,
- $\mathcal{S}_n$ is the set of the internal edges of the triangles of $\mathcal{T}_n$.



Standard Convergence Results

For H_n^{max} small enough:

$$|\lambda - \lambda_n| \;\leq\; C_{spec}^2 (H_n^{max})^{2s},$$

and

$$|||u - \alpha_n u_n|||_\Omega \;\leq\; C_{spec} (H_n^{max})^{s},$$

where s depends on the regularity if the eigenfunction.

Strang & Fix (1973), Babuška & Osborn (1991)

Residual

Definition (Jump)

$$[g]_f(x) := \left(\lim_{\substack{\tilde{x}\in\tau_1(f)\\ \tilde{x}\to x}} g(\tilde{x}) - \lim_{\substack{\tilde{x}\in\tau_2(f)\\ \tilde{x}\to x}} g(\tilde{x}) \right), \quad \text{with } x \in f.$$

Definition (Residual)

$$R_I(u,\lambda)(x) := \left(\nabla\cdot\mathcal{A}\nabla u + \lambda\mathcal{B}u\right)(x), \quad \text{with } x \in int(\tau), \quad \tau\in\mathcal{T}_n,$$

$$R_F(u)(x) := \left[\vec{n}_f\cdot\mathcal{A}\nabla u\right]_f(x), \quad \text{with } x\in int(f), \quad f\in\mathcal{S}_n.$$

$$\eta_n := \left\{ \sum_{\tau\in\mathcal{T}_n} H_\tau^2 \|R_I(u_n,\lambda_n)\|_{0,\tau}^2 + \sum_{f\in\mathcal{S}_n} H_f \|R_F(u_n)\|_{0,f}^2 \right\}^{1/2},$$

Reliability for Eigenfunctions

Theorem (Reliability for eigenfunctions)

Let (λ, u) be a simple eigenvalue of the continuous problem and let (λ_n, u_n) be the correspondent computed eigenpair. Then we have for $e_n = u - \alpha_n u_n$ that

$$|||e_n|||_\Omega \leq C\ \eta_n\ +\ G_n,$$

where

$$G_n = \frac{1}{2}(\lambda + \lambda_n)\frac{b(e_n, e_n)}{|||e_n|||_\Omega}.$$

Reliability for Eigenvalues

Theorem (Reliability for eigenvalues)

Let (λ, u) be a simple eigenvalue of the continuous problem and let (λ_n, u_n) be the correspondent computed eigenpair. Then we have for $e_n = u - \alpha_n u_n$ that

$$|\lambda_n - \lambda| \leq C' \, \eta_n^2 \, + \, G_n',$$

where

$$G_n' = \eta_n \frac{1}{2}(\lambda + \lambda_n) \frac{b(e_n, e_n)}{|||e_n|||_\Omega} \, + \, \frac{1}{2}(\lambda - \lambda_n) b(e_n, e_n) \, .$$

Efficiency

Theorem (Efficiency)

Let (λ, u) be a simple eigenvalue of the continuous problem and let (λ_n, u_n) be the corresponding computed eigenpairs. Then we have that the global residual estimator is bounded by the energy norm of the error as:

$$\eta_n \leq C''' \, |||e_n|||_\Omega \; + \; \|H_\tau(\lambda_n \alpha_n u_n - \lambda u)\|_{0,\mathcal{B},\Omega}.$$

Properties for H_n^{max} small enough:

$$C'''^{-1} \, \eta_n \;\leq\; |||u - \alpha_n u_n|||_\Omega \;\leq C \; \eta_n \; .$$

Constants C and C''' are independent of H_n^{max}.

Marking Strategy 1

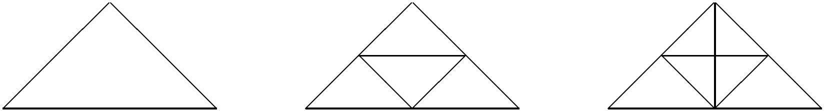
Set the parameter $0 < \theta < 1$:
mark the edges (faces) in a minimal subset $\hat{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that

$$\left(\sum_{\tau \in \hat{\mathcal{T}}_n} \eta_{\tau,n}^2 \right)^{1/2} \geq \theta \eta_n \ ,$$

where

$$\eta_{\tau,n}^2 \ := \ H_\tau^2 \| R_I(u_n, \lambda_n) \|_{0,\tau}^2 \ + \frac{1}{2} \sum_{f \in \mathcal{S}_\tau} H_f \| R_F(u_n) \|_{0,f}^2 \ .$$

Bisection5



1. Split all edges
2. Split one of the new edges

PROS:

1. A new node on each edge
2. A new node in the interior of the element

Oscillations

Oscillations:

$$osc(v_n, \mathcal{T}_n) := \left(\sum_{\tau \in \mathcal{T}_n} \|H_\tau(v_n - P_n v_n)\|_\tau^2 \right)^{1/2},$$

where $(P_n v_n)|_\tau := \frac{1}{|\tau|} \int_\tau v_n$.

P. Morin, R. H. Nochetto, and K. G. Siebert (2000)
Data oscillation and convergence of adaptive FEM.
SIAM J. Numer. Anal. 38, 466-488.

Marking Strategy 2

Set the parameter $0 < \tilde{\theta} < 1$:
mark the sides in a minimal subset $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that

$$osc(u_n, \tilde{\mathcal{T}}_n) \geq \tilde{\theta}\, osc(u_n, \mathcal{T}_n).$$

Then we take the union of $\hat{\mathcal{T}}_n \cup \tilde{\mathcal{T}}_n$ and we refine all the elements in the union.

Adaptivity Algorithm

1. Require $0 < \theta < 1$, $0 < \tilde{\theta} < 1$ and $\mathcal{T}_0$
2. Loop
3. Compute (λ_n, u_n) on $\mathcal{T}_n$
4. Marking strategy 1
5. Marking strategy 2
6. Refine the mesh
7. End Loop

Convergence for Adaptive FEMs

Convergence for Adaptive Finite Element Methods for Linear Boundary Value Problems:
Dörfler (1996), Morin, Nochetto & Siebert (2000,2002), Karakashian & Pascal (2003), Mekchay & Nochetto (2005), Mommer & Stevenson (2006), Morin, Siebert & Veeser (2007), Cascon, Kreuzer Nochetto & Siebert (2008), ...

Convergence for Adaptive Finite Element Methods for Eigenvalue Problems:
G.& Graham (2009), Dai, Xu & Zhou (2008), Carstensen & Gedicke (2009), Garau, Morin & Zuppa (2009)

Convergence

Theorem (Convergence Result)

Provided that λ is a simple eigenvalue and that on the initial mesh H_0^{max} is small enough, there exists a constant $p \in (0,1)$ and constants C_0, C_1 such that the recursive application of the algorithm yields a convergent sequence of approximate eigenvalues and eigenvectors, with the property:

$$|||u - \alpha_n u_n \,|||_\Omega \leq C_0 p^n \;,$$

$$|\lambda - \lambda_n| \leq C_0^2 p^{2n} \;,$$

and

$$osc(\lambda_n u_n, \mathcal{T}_n) \leq C_1 p^n .$$

Error Reduction

Theorem (Error Reduction)

For each $\theta \in (0,1)$, exists a sufficiently fine mesh threshold $H_n^{\max}$ and constants $\mu > 0$ and $\rho \in (0,1)$ such that: For any $\epsilon > 0$ then inequality

$$osc(u_n, \mathcal{T}_n) \leq \mu\epsilon \ ,$$

implies either

$$|||u - \alpha_n u_n|||_\Omega \leq \epsilon \ ,$$

or

$$|||u - \alpha_{n+1} u_{n+1}|||_\Omega \leq \rho \, |||u - \alpha_n u_n|||_\Omega \ .$$

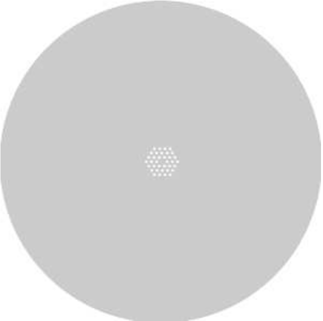
Oscillations Reduction

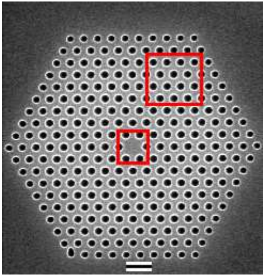
Theorem (Oscillations Reduction)

There exists a constant $\tilde{\rho} \in (0,1)$ *such that:*

$$osc(u_{n+1}, \mathcal{T}_{n+1}) \leq \tilde{\rho} osc(u_n, \mathcal{T}_n) \; + \; C(H_n^{\max})^2 \, |||u - \alpha_n u_n \, |||_\Omega \; .$$

Photonic Crystal Fibers (PCFs)





Applications: communications, filters, lasers, switchers

Figotin & Klein (1998), Cox & Dobson (1999), Dobson (1999), Sakoda (2001), Kuchment (2001), Figotin & Goren (2001), Johnson & Joannopoulos (2002), Ammari & Santosa (2004), Joannopoulos, Johnson, Winn & Meade (2008),...

S. G.
Convergence of Adaptive Finite Element Methods for Elliptic Eigenvalue Problems with Applications to Photonic Crystals.
Ph.D. Thesis , University of Bath (2008)

Variational Formulation (e.g. TE)

$$a_\kappa(u,v) := \int_\Omega (\nabla + i\vec{\kappa})u \cdot \mathcal{A}(\nabla - i\vec{\kappa})\bar{v},$$

$$(u,v)_{0,\Omega} := \int_\Omega u\bar{v}.$$

Continuous Problem:

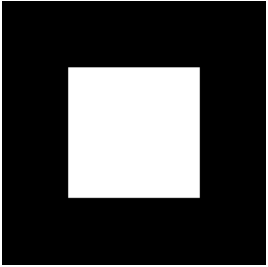
seek eigenpairs of the form $(\lambda, u) \in \mathbb{R} \times H^1_\pi(\Omega)$*, with* $\|u\|_{0,\Omega} = 1$ *such that*

$$a_\kappa(u,v) = \lambda(u,v)_{0,\Omega} \quad \text{for all } v \in H^1_\pi(\Omega).$$

Energy Norm:

$$|||u|||^2_{\kappa,\mathcal{A},\Omega} := a_\kappa(u,u).$$

Periodic Structure (I)



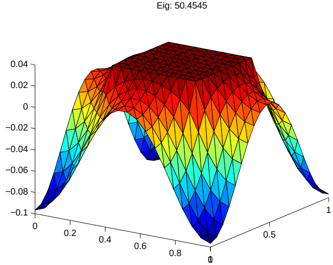


Figure: A cell of the periodic structure with $\mathcal{A} = 1$ outside and $\mathcal{A} = 10000$ inside and a picture of the eigenfunction corresponding to the second smallest eigenvalue for quasimomentum $(0, 0)$

Periodic Structure (II)

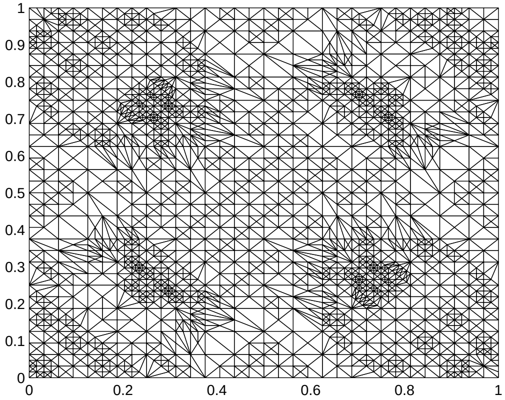


Figure: An adapted mesh for the periodic structure with $\theta = \tilde{\theta} = 0.8$.

Periodic Structure (III)

Uniform			Adaptive		
$\lvert\lambda - \lambda_n\rvert$	N	β	$\lvert\lambda - \lambda_n\rvert$	N	β
1.3556	400	-	1.3556	400	-
0.4567	1600	0.7848	0.6362	903	0.9291
0.1596	6400	0.7584	0.2124	2690	1.0048
0.0563	25600	0.7516	0.1237	5495	0.7571
0.01489	102400	0.7874	0.0405	15709	1.0640

Table: Comparison between uniform and adaptive refinement (with $\theta = \tilde{\theta} = 0.8$) for the second smallest eigenvalue of the TE mode problem with quasimomentum $(0, 0)$.

$$|\lambda - \lambda_n| = \mathcal{O}(N^{-\beta}), \quad N = \#DOF.$$

The University of Nottingham

Defect modes (I)

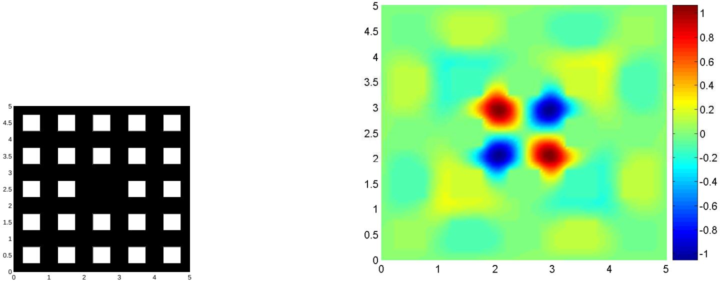


Figure: The structure of the supercell and the trapped mode.

Defect modes (II)

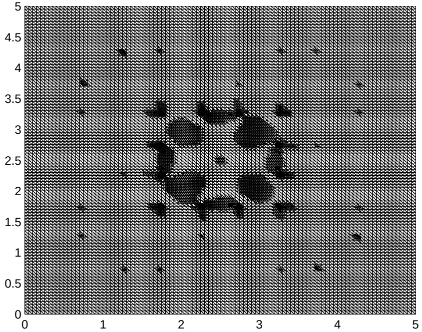


Figure: An adapted mesh for the periodic structure with $\theta = \tilde{\theta} = 0.8$.

Refinement in the interior and at the corners of the inclusions.

Defect modes (III)

Uniform			Adaptive		
$\lvert\lambda - \lambda_n\rvert$	N	β	$\lvert\lambda - \lambda_n\rvert$	N	β
0.5858	10000	-	0.5858	10000	-
0.1966	40000	0.7876	0.1225	20506	2.1791
0.0653	160000	0.7951	0.0579	44548	0.9659
0.0188	640000	0.8982	0.0078	220308	1.2541

Table: Comparison between uniform and adaptive refinement (with $\theta = \tilde{\theta} = 0.8$) for a trapped mode in the supercell for TE mode problem.

$$|\lambda - \lambda_n| = \mathcal{O}(N^{-\beta}), \quad N = \#DOF.$$

Summary

- We prove the convergence of an adaptive finite element method for elliptic eigenvalue problems,
- The proof exploits reduction results for error and oscillations,
- Consequences:
  - The computed approximated eigenpairs are approximation of true eigenpairs,
  - For any tolerance $\mathrm{tol} > 0$ the adaptive algorithm will end after a finite number of iterations.