

High-Order/ hp -Adaptive Multilevel Discontinuous Galerkin Methods

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Joint work with
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www.AptoFEM.com

Reading 2014

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- Aims

2 DGCFE Method

- An overview
- Finite element space
- SIP DGCFE method

3 Adaptivity

- Energy Norm Error Estimator
- Goal-oriented Error Estimator

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We want a discontinuous Galerkin (DG) method such that:

- 1 The number of DOFs is independent of how complicated is the domain
- 2 A method easy to extend to high-orders.
- 3 A method such that all adaptive techniques we already use are easy to use with
- 4 Any new DOF inserted by adaptivity is useful to improve the accuracy and not wasted to describe the domain
- 5 A preconditioner working on a variety of problems

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DGCFE (Discontinuous Galerkin Composite Finite Element) method:



P. Antonietti, S. G. and P. Houston

hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains

SISC, 35(3), 1417–1439.

It is an extension of CFE for continuous Galerkin:



W. Hackbusch and S.A. Sauter

Composite finite elements for the approximation of PDEs on domains with complicated micro-structures

Numer. Math., 75, 447–472, 1997.



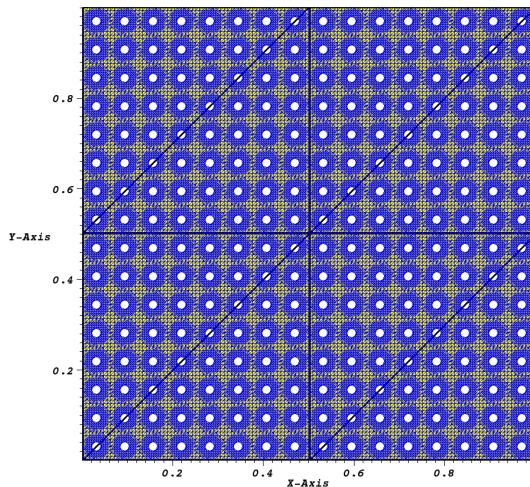
W. Hackbusch and S.A. Sauter

Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results

Comput. Visual Sci., 1, 15–25, 1997.

Two meshes:

- Mesh $\mathcal{T}_{h_\ell}$ is a partition of the domain Ω and describes all the details in the domain.
- A coarser mesh $\mathcal{T}_{\text{CFE}}$ which is too coarse to describe the details in the domain Ω .



Transferring information from the fine level to the coarse level:

- 1 The geometrical details are not “stored” in the coarse mesh $\mathcal{T}_{\text{CFE}}$ but in the finite element basis functions on the coarse level.
- 2 Boundary conditions are imposed weakly.
- 3 The mesh $\mathcal{T}_{\text{CFE}}$ and the correspondent finite element space $V(\mathcal{T}_{\text{CFE}}, p)$ are used to set the size and the sparsity of the linear system.
- 4 The mesh $\mathcal{T}_{h_\ell}$ and the correspondent finite element space $V(\mathcal{T}_{h_\ell}, p)$ are used to compute the entries of the linear system.

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On each CFE element τ_{CFE} we set up a polynomial space of order p . The domain of each element τ_{CFE} is the union of its children. Also on the children elements of τ_{CFE} we set up polynomial spaces of same order p . In this way the basis functions of τ_{CFE} can be expressed using the basis functions of the children:

$$\phi_{\text{CFE},i} := \sum_{j=1, \dots, \dim(V(\mathcal{T}_{h_\ell}, p))} \alpha_{i,j} \phi_{h_\ell,j} .$$

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Theorem

Let $\Omega \subset \mathbb{R}^d$ be a bounded polyhedral domain, and let $\mathcal{T}_{\text{CFE}} = \{\kappa\}$ be a subdivision of Ω . Let $u_h \in V(\mathcal{T}_{\text{CFE}}, p)$ be the composite discontinuous Galerkin approximation to u and suppose that $u|_{\kappa} \in H^{k_{\kappa}}(\kappa)$ for each $\kappa \in \mathcal{T}_{\text{CFE}}$ for integers $k_{\kappa} \geq 1$. Then, the following error bound holds

$$||| u - u_h |||_{\text{DG}}^2 \leq C \sum_{\kappa \in \mathcal{T}_{\text{CFE}}} \frac{h_{\kappa}^{2s_{\kappa}}}{h_F^2} \frac{1}{p_{\kappa}^{2k_{\kappa}-3}} \|\mathfrak{E}\tilde{u}\|_{H^{k_{\kappa}}(\hat{\kappa})}^2,$$

for any integers s_{κ} , $1 \leq s_{\kappa} \leq \min(p_{\kappa} + 1, k_{\kappa})$, and $p_{\kappa} \geq 1$. Here, C is a positive constant that depends only on the dimension d .

$$||| v |||_{\text{DG}}^2 = \sum_{\kappa \in \mathcal{T}_{\text{CFE}}} \|\nabla v\|_{L_2(\kappa)}^2 + \sum_{F \in \mathcal{F}_{\text{CFE}}^{\mathcal{I}} \cup \mathcal{F}_{\text{CFE}}^{\mathcal{B}}} \|\sigma^{1/2} \llbracket v \rrbracket\|_{L_2(F)}^2.$$

$$-\Delta u = f \text{ in } \Omega,$$

$$u = g \text{ on } \partial\Omega.$$

Solution:

$$u = \sin(\pi x) \cos(\pi y).$$

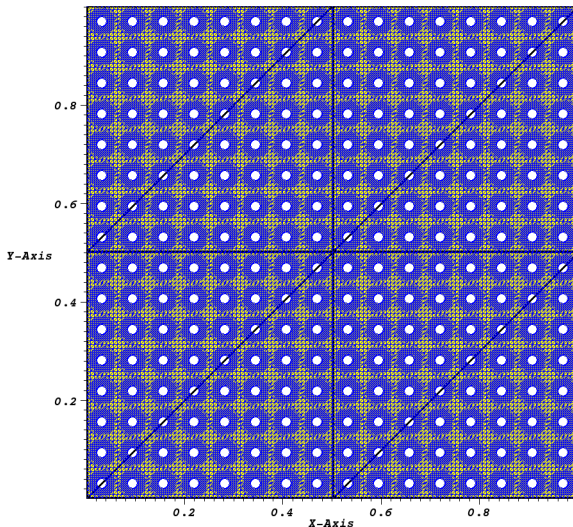
Ω has micro-structures.

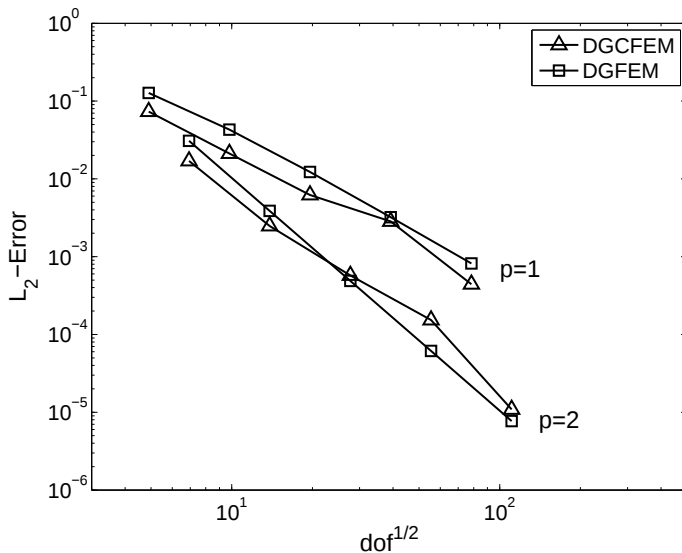


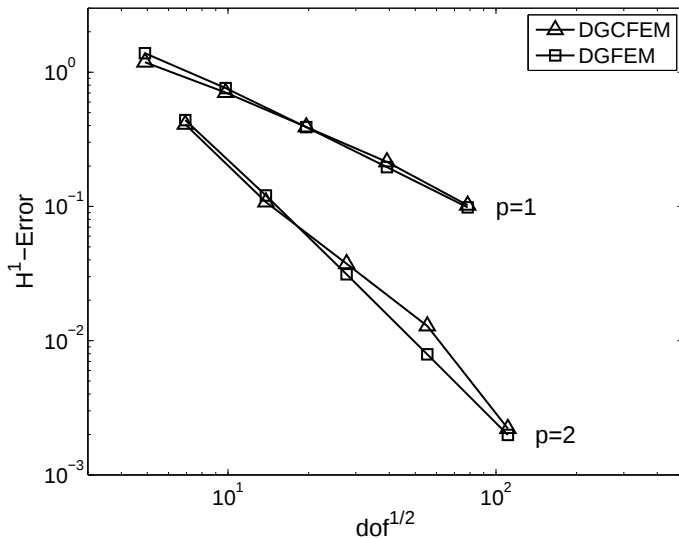
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hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains

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Cons:

- ❶ **Not Flexible:** only the norm of the error can be predicted
- ❷ **No Sharp Bound:** there is a hidden constant

Pros:

- ❶ **Cheap:** easy to compute

$$||| u - u_h |||_{\text{DG}} \leq C \left(\sum_{\kappa \in \mathcal{T}_{\text{CFE}}} (\eta_{\kappa}^2 + \mathcal{O}_{\kappa}^2) \right)^{\frac{1}{2}}, \quad (1)$$

where the local error indicators η_{κ} , $\kappa \in \mathcal{T}_{\text{CFE}}$, are defined by

$$\begin{aligned} \eta_{\kappa}^2 &= h_{\kappa}^2 p_{\kappa}^{-2} \|\Pi f + \Delta u_h\|_{L_2(\kappa)}^2 \\ &+ \sum_{F \subset \partial \kappa \setminus \partial \Omega} h_{\kappa}^2 h_F^{-1} p_{\kappa}^{-1} \|[\![\nabla u_h]\!]\|_{L_2(F)}^2 + \sigma h_{\kappa}^2 h_F^{-2} p_{\kappa} \|[\![u_h]\!]\|_{L_2(\partial \kappa)}^2 \end{aligned} \quad (2)$$

and the data oscillation term $\mathcal{O}_{\kappa}$ is given by

$$\mathcal{O}_{\kappa} = h_{\kappa}^2 p_{\kappa}^{-2} \|f - \Pi f\|_{L_2(\kappa)}^2.$$



S. G. and P. Houston

hp-Adaptive Composite Discontinuous Galerkin Methods for Elliptic Problems on Complicated Domains

Numerical Methods for Partial Differential Equations, Accepted.

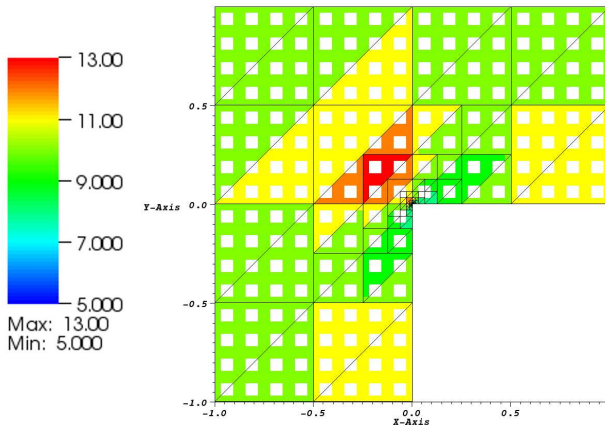


Figure: 16th adapted mesh to approximate the solution $u = r^{2/3} \sin(2\varphi/3)$.

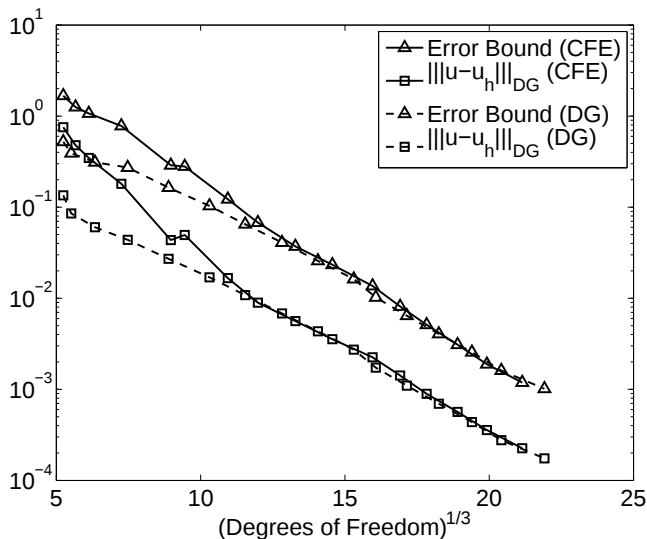


Figure: Convergence

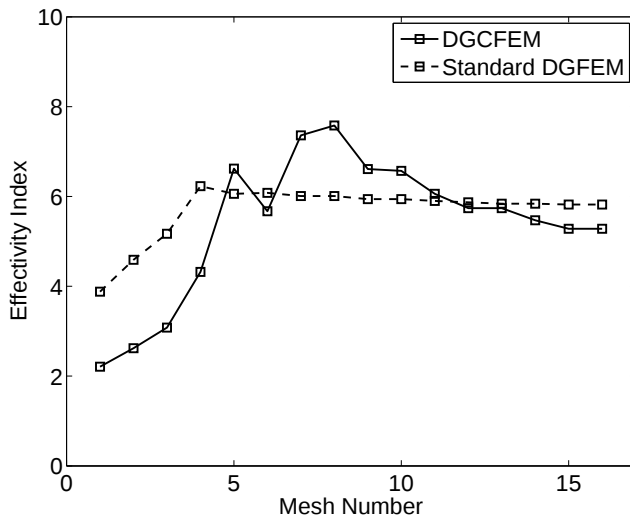


Figure: Effectivity

Plateau example

The rhs is defined as $f = 0$ in the subdomain $[0.25, 0.75]^2$ and $f = 1$ everywhere else.

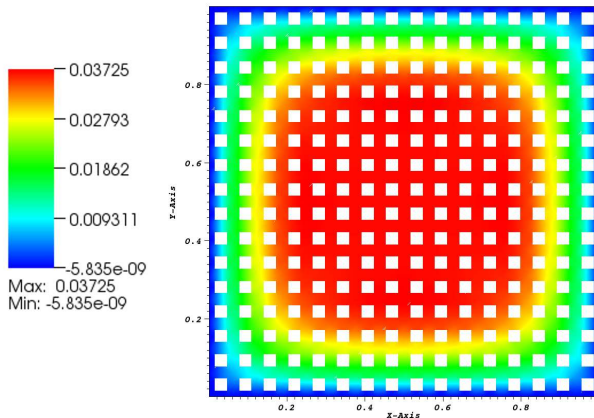


Figure: Solution.

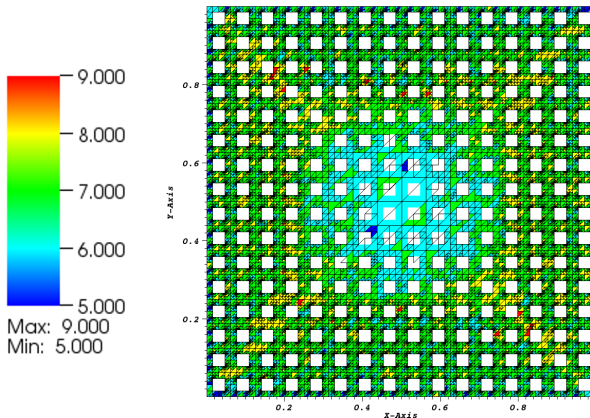


Figure: 18th adapted mesh.

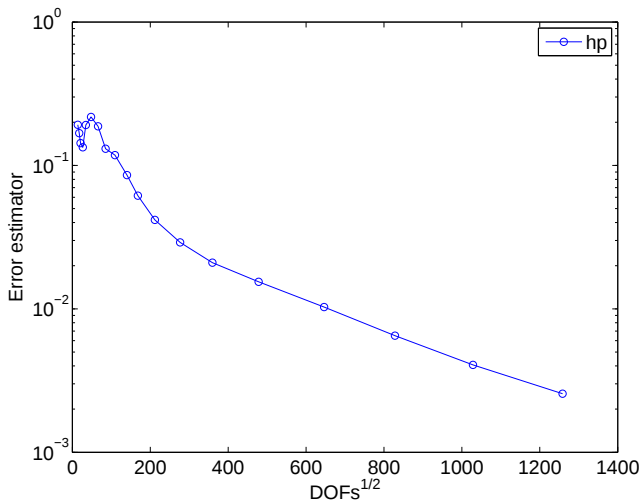


Figure: Convergence

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Cons:

- ① **Expensive:** a dual problem should be solved

Pros:

- ① **Flexible:** the quantity of interest can be chosen
- ② **Sharp Bound:** the upper bound has constant 1

$$|J(u) - J(u_{hp})| \lesssim \sum_{\kappa \in \mathcal{T}_h} |\tilde{\eta}_{\kappa}(u_{hp}, \tilde{z}_{hp})| .$$

- ③ **Accurate:** it is possible to compute an accurate estimation of the error in the quantity of interest

$$J(u) - J(u_{hp}) \approx \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_{\kappa}(u_{hp}, \tilde{z}_{hp}) ,$$

- ④ **Improved Accuracy:** it is possible to improve the accuracy of the quantity of interest

$$J(u) \approx J(u_{hp}) + \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_{\kappa}(u_{hp}, \tilde{z}_{hp}) ,$$

$$-\frac{1}{Re}\nabla^2\mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) + \nabla p = \mathbf{0}, \quad \text{in } \Omega, \quad (3)$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \quad (4)$$

with boundary conditions

$$\mathbf{u} = \mathbf{g}_D \quad \text{on } \partial\Omega_D, \quad \frac{1}{Re}\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = 0 \quad \text{on } \partial\Omega_N, \quad (5)$$



S.G. and P. Houston

Goal–Oriented Adaptive Composite Discontinuous Galerkin Methods
for Incompressible Flows

Journal of Computational and Applied Mathematics (2014) 270,
32–42.

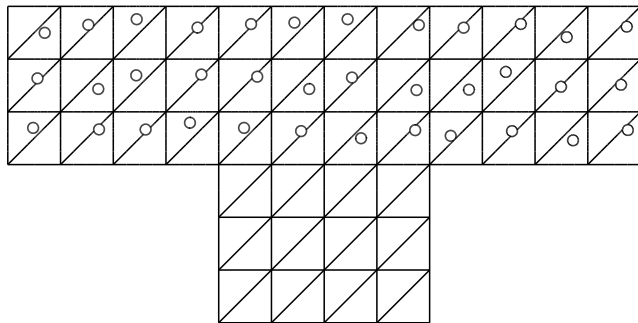


Figure: First Mesh

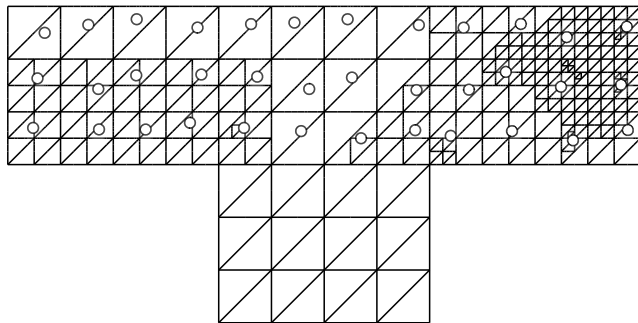


Figure: Fourth Mesh

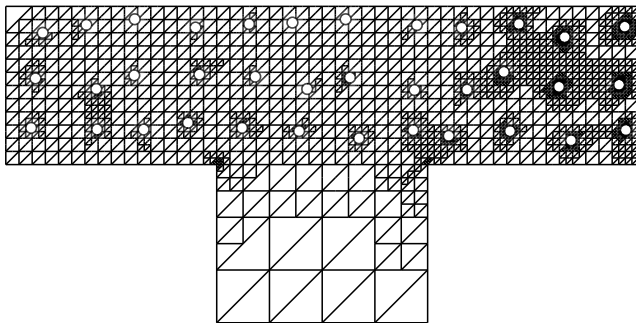


Figure: Eighth Mesh

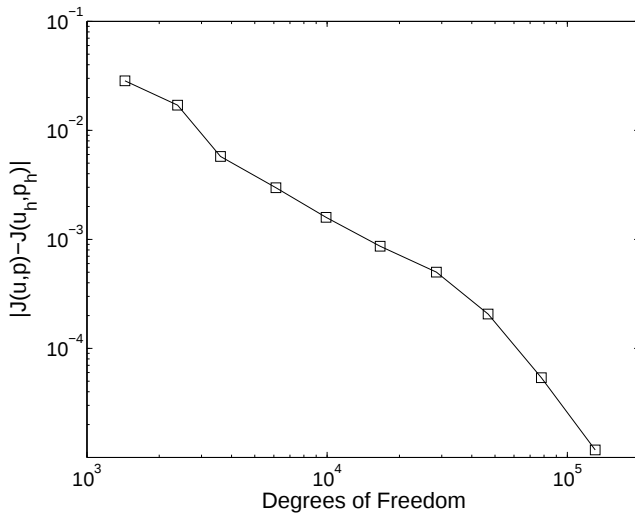


Figure: Convergence

No of Eles	No of Dofs	$J(\mathbf{u}, p) - J(\mathbf{u}_h, p_h)$	$\sum_{\kappa \in \mathcal{T}_{\text{GFE}}} \eta_{\kappa}$	θ
96	1440	-2.849E-2	-2.534E-2	0.89
159	2385	-1.702E-2	-1.430E-2	0.84
240	3600	-5.755E-3	-3.419E-3	0.59
408	6120	-2.974E-3	-1.554E-3	0.52
660	9900	-1.592E-3	-7.969E-4	0.50
1108	16620	-8.644E-4	-3.853E-4	0.45
1901	28515	-5.008E-4	-2.842E-4	0.57
3118	46770	-2.068E-4	-1.468E-4	0.71
5196	77940	-5.390E-5	-4.716E-5	0.87
8708	130620	-1.172E-5	-1.172E-5	1.00

Table: Adaptive algorithm.