Convergence of adaptive FEM for elliptic eigenvalue problems

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Model Problem

Let Ω be a convex polygonal domain bounded in $\mathbb{R}^2$
Problem: seek eigenpairs (λ, u) of the problem

$$\begin{cases} -\triangle u &= \lambda u \qquad \text{in } \Omega, \\ \quad u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Variational Problem: seek eigenpairs $(\lambda, u) \in \mathbb{R} \times H_0^1(\Omega)$ such that

$$a(u, v) \;=\; \lambda(u, v)_{0,\Omega} \quad \text{for all } v \in H_0^1(\Omega),$$

where

$$\begin{aligned} a(u, v) &:= \int_\Omega \nabla u \cdot \nabla v \, dx, \\ |||v|||_\Omega &:= a(v, v)^{1/2}. \end{aligned}$$

Meshes and Discrete Problems

Define:

- $\mathcal{T}_n$ conforming and shape regular triangulation of Ω
- $\mathcal{S}_n$ is the set of the edges of the triangles of $\mathcal{T}_n$,
- V_n space of piecewise linear functions over $\mathcal{T}_n$.

Problem: seek eigenpairs $(\lambda_n, u_n) \in \mathbb{R} \times V_n$ such that

$$a(u_n, v_n) = \lambda_n (u_n, v_n)_{0,\Omega} \quad \text{for all } v_n \in V_n.$$

Standard convergence results for uniform refinement

For H_n^{max} small enough:

$$|\lambda - \lambda_n| \;\leq\; C_{spec}^2 (H_n^{max})^2,$$

and

$$|||u - u_n|||_\Omega \;\leq\; C_{spec} H_n^{max},$$

[Strang and Fix, 1973], [Babuška and Osborn, 1991]

A Posteriori Error Estimator

$$\eta_n := \left(\sum_{\tau \in \mathcal{T}_n} \| H_\tau \lambda_n u_n \|_{0,\tau}^2 \; + \; \sum_{S \in \mathcal{S}_n} \| H_S^{1/2} [\nabla u_n] \|_{0,S}^2 \right)^{1/2},$$

Properties:

1. $|||u - u_n|||_\Omega \; \leq C_{rel} \eta_n \; + G_n$ (Reliability)
2. $|\lambda - \lambda_n| \; \leq \; C_{rel}^2 {\eta_n}^2 \; + F_n$ (Reliability)
3. $\eta_n \; \leq \; C_{eff} |||u - u_n|||_\Omega \; + E_n$ (Efficiency)

Constants C_{rel} and C_{eff} are independent of H_n^{max} and G_n, F_n and E_n are h.o.t.

Marking Strategy 1

Set the parameter $0 < \theta < 1$:
mark the sides in a minimal subset $\hat{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that

$$\left(\sum_{\tau \in \hat{\mathcal{T}}_n} \eta_{\tau,n}^2 \right)^{1/2} \geq \theta \eta_n,$$

where

$$\eta_{\tau,n}^2 := \|H_\tau \lambda_n u_n\|_{0,\tau}^2 + \sum_{S \in \partial\tau} \frac{1}{2} \|H_S^{1/2} [\nabla u_n]\|_{0,S}^2.$$

Oscillations

Oscillations: [Morin et al., 2000]

$$osc(v_n, \mathcal{T}_n) := \left(\sum_{\tau \in \mathcal{T}_n} \| H_\tau (v_n - P_n v_n) \|_\tau^2 \right)^{1/2},$$

where $(P_n v_n)|_\tau := \frac{1}{|\tau|} \int_\tau v_n$

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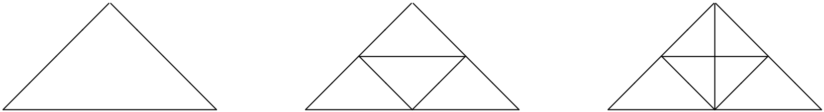
Marking Strategy 2

Set the parameter $0 < \tilde{\theta} < 1$:
mark the sides in a minimal subset $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that

$$osc(u_n, \tilde{\mathcal{T}}_n) \geq \tilde{\theta}\, osc(u_n, \mathcal{T}_n).$$

Then we take the union of $\hat{\mathcal{T}}_n \cup \tilde{\mathcal{T}}_n$ and we refine all the elements in the union.

Bisection5



1. Split any edge
2. Split one of the new edge

PROS:

1. A new node on each edge
2. A new node in the interior of the element

Mesh Adaptivity Algorithm

1. Require $0 < \theta < 1$, $0 < \tilde{\theta} < 1$ and $\mathcal{T}_0$
2. Loop
3. Compute (λ_n, u_n) on $\mathcal{T}_n$
4. Marking strategy 1
5. Marking strategy 2
6. Refine the mesh
7. End Loop

Error Reduction

Theorem (Error reduction)

For each $\theta \in (0,1)$, exists a sufficiently fine mesh threshold H_n^{max} and constants $\mu > 0$ and $\alpha \in (0,1)$, with the following property. For any $\varepsilon > 0$ the inequality

$$osc(\lambda_n u_n, \mathcal{T}_n) \leq \mu\varepsilon,$$

implies either $|||u - u_n\,|||_\Omega \leq \varepsilon$ or

$$|||u - u_{n+1}\,|||_\Omega \leq \alpha\,|||u - u_n\,|||_\Omega.$$

Remark:

- The decay of oscillations trigs the convergence in the energy norm.

Oscillations Reduction

Theorem

Let $\mathcal{T}_n$ to be a mesh and $\mathcal{T}_{n+1}$ a refinement of the first mesh. Let also (λ_n, u_n) and (λ_{n+1}, u_{n+1}) be to approximations of the same true eigenpair (λ, u) computed on the two meshes respectively. Then a constant $\tilde{\alpha} \in (0, 1)$ exists such that

$$osc(\lambda_{n+1} u_{n+1}, \mathcal{T}_{n+1}) \leq \tilde{\alpha}\, osc(\lambda_n u_n, \mathcal{T}_n) + C \lambda_n |||u - u_n\ |||_\Omega .$$

Remark:

The convergence in the energy norm trigs the decay of oscillations.

Cross Feedback

- The decay of oscillations trigs the convergence in the energy norm,
- The convergence trigs the decay of oscillations,

So:

We have a feedback loop between error and oscillations.

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Convergence

Theorem (Convergence Result)

Provided the initial mesh is chosen so that H_0^{max} is small enough, there exists a constant $p \in (0,1)$ and constants C_0, C_1 and $q > 1$ such that the recursive application of the algorithm yields a convergent sequence of approximate eigenvalues and eigenvectors, with the property:

$$|||u - u_n|||_\Omega \leq C_0 q p^n,$$

$$|\lambda - \lambda_n| \leq C_0^2 q^2 p^{2n},$$

and

$$osc(\lambda_n u_n, \mathcal{T}_n) \leq C_1 p^n.$$

Adaptivity for smooth problem (I)

	$\theta = \tilde{\theta} = 0.5$			$\theta = \tilde{\theta} = 0.8$		
Iteration	$\lvert\lambda - \lambda_n\rvert$	DOFs	β	$\lvert\lambda - \lambda_n\rvert$	DOFs	β
1	0.1350	400	-	0.1350	400	-
2	0.1177	954	0.1581	0.0529	1989	0.5839
3	0.0779	1564	0.8349	0.0176	5205	1.1407
4	0.0501	1977	1.8788	0.0073	15980	0.7877
5	0.0351	2634	1.2383	0.0024	48434	0.9836
6	0.0176	4004	0.7885	0.0009	122699	1.0673
7	0.0121	6588	0.7217	0.0003	312591	1.0083

Table: Comparison of the reduction of the error and DOFs of the adaptive method for the first eigenvalue for the Laplace problem.

Adaptivity for smooth problem (II)

	$\theta = \tilde{\theta} = 0.5$			$\theta = \tilde{\theta} = 0.8$		
Iteration	$\lvert\lambda - \lambda_n\rvert$	DOFs	β	$\lvert\lambda - \lambda_n\rvert$	DOFs	β
1	2.1439	400	-	2.1439	400	-
2	1.8280	1016	0.1658	0.7603	2039	0.6365
3	1.0850	1636	1.1662	0.2439	6793	0.9447
4	0.7792	12254	1.0331	0.0917	18717	0.9652
5	0.4936	3067	1.4826	0.0331	54113	0.9583
6	0.3484	4681	0.8240	0.0120	146056	1.0181
7	0.2578	7321	0.6730	0.0046	382024	0.9970

Table: Comparison of the reduction of the error and DOFs of the adaptive method for the fourth eigenvalue for the Laplace problem.

Adaptivity for non-smooth problem (I)

$$a(u,v) = \int_\Omega \nabla u(x)^T \mathcal{A}(x) \nabla v(x) dx.$$

	$\theta = \tilde{\theta} = 0.5$			$\theta = \tilde{\theta} = 0.8$		
Iteration	$\lvert\lambda - \lambda_n\rvert$	DOFs	β	$\lvert\lambda - \lambda_n\rvert$	DOFs	β
1	1.1071	81	-	1.1071	81	-
2	0.7959	216	0.3364	0.4214	362	0.6452
3	0.6075	301	0.8139	0.1955	1153	0.6628
4	0.4168	437	1.0108	0.0789	2811	1.0174
5	0.2750	643	1.0762	0.0335	6534	1.0151
6	0.1989	954	0.8212	0.0172	14059	0.8687
7	0.1236	1459	1.1186	0.0066	28341	1.3621
8	0.0935	2117	0.7504	0.0033	60148	0.9123

Table: Comparison of the reduction of the error and DOFs of the adaptive method for the first eigenvalue for the problem with discontinuous coefficients.

Adaptivity for non-smooth problem (II)

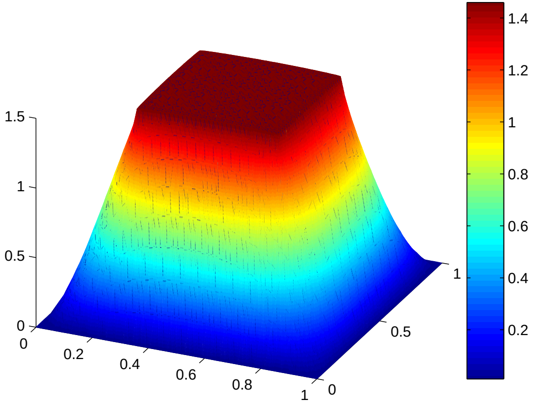

Figure: An approximation of the eigenfunction corresponding to the smallest eigenvalue.

Adaptivity for non-smooth problem (III)

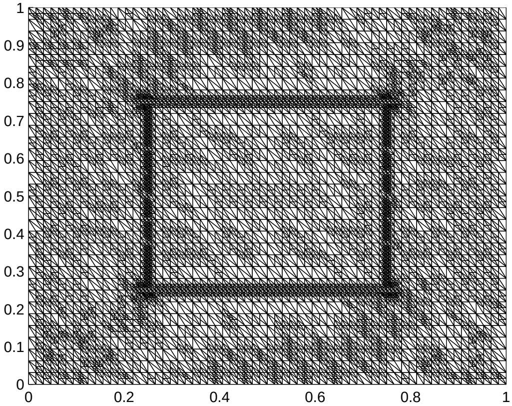

Figure: A refined mesh from the adaptive method corresponding to the first eigenvalue of the problem with discontinuous coefficients.

Future work:

- Rid of the oscillations and the bisection5;
- Extend the result to more complicate operators:

$$a(u,v) = \int_\Omega (\nabla + i\boldsymbol{\kappa})u(x)^T \mathcal{A}(x)(\nabla - i\boldsymbol{\kappa})\bar{v}(x)dx.$$

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I. Babuška and J. Osborn.
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P. Morin, R. H. Nocetto and K. G. Siebert.
Data Oscillation and Convergence of Adaptive FEM.
SIAM J. Numer. Anal., 38:466-488, 2000

[Proof]

Proof of Convergence (I)

Lemma

For H_n^{max} small enough and taking a computed eigenpair (λ_n, u_n) converging to (λ, u); we have that there exists a constant $q > 1$ such that on any mesh $\mathcal{T}_m$ with $m > n$, which is a refinement of $\mathcal{T}_n$, the corresponding computed eigenpair (λ_m, u_m) satisfies:

$$|||u - u_m\ |||_\Omega \leq q\ |||u - u_n\ |||_\Omega.$$

Remarks:

- refine the mesh could increase the error in nonlinear problems,
- the error does not blow up refining the mesh.

Proof of Convergence (II)

We prove only the statement

$$|||u - u_n|||_\Omega \leq C_0 q p^n,$$

so we suppose that

$$osc(\lambda_n, u_n, \mathcal{T}_n) \leq C_1 p^n.$$

Let's choose p and C_1 such that

$$\max\{\alpha, \tilde{\alpha}\} < p < 1,$$

and

$$C_1 = osc(\lambda_0, u_0, \mathcal{T}_0).$$

Let's define C_0 as

$$C_0 = \max\{\mu^{-1} p^{-1} C_1, |||u - u_0|||_\Omega\}.$$

Proof of Convergence (III)

Initial Step:

$$|||u - u_0\,|||_\Omega \leq C_0 \leq C_0 q,$$

Proof of Convergence (IV)

Induction Step:
We suppose that

$$|||u - u_n|||_\Omega \leq C_0 q p^n.$$

If

$$|||u - u_n|||_\Omega \leq C_0 p^{n+1},$$

then

$$|||u - u_{n+1}|||_\Omega \leq q\, |||u - u_n|||_\Omega \leq q C_0 p^{n+1}.$$

Proof of Convergence (V)

Induction Step:
We suppose that

$$|||u - u_n|||_\Omega \leq C_0 q p^n.$$

If

$$|||u - u_n|||_\Omega > C_0 p^{n+1},$$

then

$$|||u - u_n|||_\Omega > C_0 p^{n+1} > \mu^{-1} C_1 p^n.$$

Proof of Convergence (VI)

In order too apply Error Reduction, we choose $\varepsilon := \mu^{-1} C_1 p^n$, and from $osc(\lambda_n, u_n, \mathcal{T}_n) \leq C_1 p^n$ we have that

$$osc(\lambda_n, u_n, \mathcal{T}_n) \leq \mu\varepsilon.$$

Since $|||u - u_n|||_\Omega > \varepsilon$ then

$$|||u - u_{n+1}|||_\Omega \leq \alpha\, |||u - u_n|||_\Omega \leq \alpha C_0 p^n \leq C_0 p^{n+1} \leq q C_0 p^{n+1}.$$