Data Oscillation and Convergence of Adaptive FEM

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P. Morin, R. H. Nocetto and K. G. Siebert.
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Outline

Introduction
- Model Problem
- Mesh Adaptivity

Old Framework
- Old Framework

New Framework
- Introduction
- Error Reduction
- Main Result

Model Problem

Let's :

- Ω a polygonal domain bounded in $\mathbb{R}^2$
- f a given function in $L^2(\Omega)$,
- A is a piecewise constant positive symmetric matrix on Ω,

Problem: seek u the solution of the problem

$$\begin{cases} -div(A\nabla u) &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Weak Form

Problem: seek $u \in H_0^1(\Omega)$ such that

$$a(u,v) = (f,v)_{0,\Omega} \quad \forall v \in H_0^1(\Omega),$$

where

$$\begin{aligned} a(u,v) &:= \int_\Omega (A\nabla u)\cdot\nabla v\,dx, \\ |||v|||_\Omega &:= \left(\int_\Omega (A\nabla v)\cdot\nabla v\,dx\right)^{1/2}, \\ (f,v)_{0,\Omega} &:= \int_\Omega fv\,dx. \end{aligned}$$

Meshes and Discrete Problems(I)

Define:

- $\mathcal{T}_H$ conforming triangulation of Ω
  $\mathcal{T}_H$ resolves the jumps of A,
- $\mathcal{S}_H$ is the set of the sides of the triangles of $\mathcal{T}_H$,
- V_H space of piecewise linear functions over $\mathcal{T}_H$.

$$V_H \subset H_0^1(\Omega) \cap C^0(\Omega) \quad \text{(conforming)}$$

Meshes and Discrete Problems(II)

Problem: seek $u_H \in V_H$ such that

$$a(u_H, v_H) = (f, v_H)_{0,\Omega} \quad \forall v_H \in V_H.$$

A Posteriori Error Estimate

$$\eta_H{}^2 := \sum_{S \in \mathcal{S}_H} \eta_{S,H}{}^2,$$

$$\eta_{S,H}{}^2 := \|H_S^{1/2} J_S\|_{0,S}^2 + \|Hf\|_{0,\Omega_S}^2,$$

$$J_S := [A \nabla u_H]_S \cdot \nu.$$

Properties:

1. $|||u - u_H|||_\Omega^2 \le C_1 \eta_H{}^2$ (Reliability)
2. $C_2 \eta_{S,H}{}^2 - C_3 \|H(f - f_H)\|_{0,\Omega_S}^2 \le |||u - u_H|||_{\Omega_S}^2$ (Local Efficiency)

All constant C_i are independent of H

Goal

Given a tolerance $\varepsilon > 0$, compute an approximated solution u_H:

$$|||u - u_H|||_\Omega \leq \varepsilon.$$

Mesh Adaptivity (Algorithm A)

1. Solve the problem for u_H
2. If $|||u - u_H|||_\Omega > \varepsilon$ Then
3. Mark the elements to be refined
4. Refine the mesh
5. Go To 1
6. End

Old Framework

Regularity: $u \in H^{1+\beta}(\Omega) \cap H_0^1(\Omega)$

$$|||u - u_H||| \leq C H_{max}^{1+\beta} |u|_{1+\beta}.$$

Remarks:

- strategy: reduce H_{max},
- refine everywhere soon or later,
- mesh adaptivity doesn't fit in this framework.

Definitions

Oscillations:

$$osc(f, \mathcal{T}_H) := \Big(\sum_{\tau \in \mathcal{T}_H} \|H_\tau(f - f_H)\|_\tau^2\Big)^{1/2}.$$

Marking Strategy: for a given $0 < \theta < 1$,

$$\left(\sum_{S \in \hat{\mathcal{S}}_H} \eta_{S,H}^2 \right)^{1/2} \geq \theta \eta_H. \tag{1}$$

Refinement by newest-vertex bisection.

Refined Mesh

Define:

- $\mathcal{T}_h$ conforming triangulation of Ω
  $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$
- V_h space of piecewise linear functions over $\mathcal{T}_h$.

$$V_h \subset H_0^1(\Omega) \cap C^0(\Omega) \quad \text{(conforming)}$$

$$V_H \subset V_h$$

Error Reduction (I)

Theorem

Let $\mathcal{T}_H$ be a triangulation of Ω, $\hat{\mathcal{T}}_H$ and $\hat{\mathcal{S}}_H$ be as defined in Marking strategy 1. Let $\mathcal{T}_h$ be the refinement of $\mathcal{T}_H$. Then there exist constants $\mu > 0$ and $0 < \alpha < 1$, such that for any $\varepsilon > 0$ if

$$osc(f, \mathcal{T}_H) \leq \mu\varepsilon,$$

then either $|||u - u_H||| \leq \varepsilon$ or the solution u_h on $\mathcal{T}_h$ satisfies

$$|||u - u_h||| \leq \alpha|||u - u_H|||.$$

Error Reduction (II)

$\mu > 0$ and $0 < \alpha < 1$ depends on:

- the minimum angle of the mesh $\mathcal{T}_H$,
- the value of θ in the Marking strategy 1,
- the continuity constant of the bilinear form $a(\cdot, \cdot)$,
- the coercivity constant of the bilinear form $a(\cdot, \cdot)$,

Remarks

- there is a condition on the initial mesh ($osc(f, \mathcal{T}_H) \leq \mu\varepsilon$),
- if all the conditions are satisfied, the prescribed tolerance ε may be met in finite steps,
- the mesh size H_{max} may not tend to 0,

Second Marking Strategy

Marking Strategy: for a given $0 < \hat{\theta} < 1$, enlarge $\hat{\mathcal{T}}_H$ such that:

$$osc(f, \hat{\mathcal{T}}_H) \geq \hat{\theta} osc(f, \mathcal{T}_H). \tag{2}$$

Oscillation Reduction

Theorem

Let $\mathcal{T}_H$ be a triangulation of Ω, $\hat{\mathcal{T}}_H$ and $\hat{\mathcal{S}}_H$ be as defined in Marking strategy 2. Let $\mathcal{T}_h$ be the refinement of $\mathcal{T}_H$.
Then there exists constant $0 < \hat{\alpha} < 1$, such that

$$osc(f, \mathcal{T}_h) \leq \hat{\alpha} osc(f, \mathcal{T}_H).$$

Mesh Adaptivity (Algorithm B)

1. Solve the problem for u_H
2. If $|||u - u_H|||_\Omega > \varepsilon$ Then
3. Marking strategy 1
4. Marking strategy 2
5. Refine the mesh
6. Go To 1
7. End

Main Result

Theorem

Let u_k be a sequence of finite element solutions produced by Algorithm B. There exist positive constants C_0, $\beta < 1$, depending on the initial mesh and the data of the problem, such that

$$|||u - u_k|||_\Omega \leq C_0 \beta^k .$$

Remarks

- the error may not decay at each single step,
- the condition on the initial mesh is only sufficient!