High-Order/hp-Adaptive Discontinuous Galerkin Finite Element Methods for Compressible Fluid Flows

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ADIGMA

Joint work with
Paul Houston (Nottingham),Manolis Georgoulis (Leicester), Edward Hall (Nottingham),
and Ralf Hartmann (DLR, Braunschweig)

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Outline

1. **Introduction**
   - Second–Order PDEs
2. **Adaptive algorithms**
   - Anisotropic h–Refinement
   - Anisotropic p–Refinement
3. **Numerical Experiments**
   - Compressible NS
   - 3D Experiments

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Abstract

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Aims

1. Construction of *High–Order DGFEMs* for a class of second–order (quasilinear) PDEs;
2. Develop the *a posteriori* error analysis and adaptive mesh design of the DGFEM approximation of target functionals of the solution based on employing *anisotropic* h–/hp–refined meshes.

- Measurement Problem: Given a user–defined tolerance TOL > 0, can we efficiently design $S_{h,p}$ such that

$$|J(\mathbf{u}) - J(\mathbf{u}_h)| \leq \mathtt{TOL}.$$

Fluid dynamics:	drag and lift coefficients.
Other examples:	point value, flux, mean value, etc.

- Applications
  - Compressible (aerodynamic) flows. ADIGMA

P. Houston (Nottingham), R. Hartmann (DLR, Braunschweig)

- Measurement Problem: Given a user–defined tolerance $\mathtt{TOL} > 0$, can we efficiently design $S_{h,p}$ such that

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Measurement Problem for Compressible

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Model Problem

- Second–Order Quasilinear System:
  Given $\Omega \subset \mathbb{R}^n$ and $\mathbf{f} \in [L_2(\Omega)]^m$, find $\mathbf{u} : \Omega \to \mathbb{R}^m$, such that

$$\mathrm{div}(\mathcal{F}^c(\mathbf{u}) - \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u})) = \mathbf{f} \quad \text{in } \Omega.$$

- Writing

$$\mathcal{F}_i^v(\mathbf{u}, \nabla \mathbf{u}) = G_{ij}(\mathbf{u}) \partial \mathbf{u} / \partial x_j, \qquad i = 1, \ldots, n,$$

  where $G_{ij}(\mathbf{u}) = \partial \mathcal{F}_i^v(\mathbf{u}, \nabla \mathbf{u}) / \partial \mathbf{u}_{x_j}$, $i, j = 1, \ldots, n$, gives

$$\frac{\partial}{\partial x_i} \left(\mathcal{F}_i^c(\mathbf{u}) - G_{ij}(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x_j} \right) = 0 \quad \text{in } \Omega.$$

- Boundary conditions, for example,

$$\mathbf{u} = \mathbf{g}_{\mathrm{D}} \quad \text{on } \partial\Omega_{\mathrm{D}}, \qquad \mathcal{F}^v(\mathbf{u}, \nabla \mathbf{u}) \cdot \mathbf{n} = \mathbf{g}_{\mathrm{N}} \text{ on } \partial\Omega_{\mathrm{N}}.$$

$$\begin{aligned}
\mathcal{N}(\mathbf{u}_h,\mathbf{v}) &:= -\int_\Omega \mathcal{F}^c(\mathbf{u}_h) : \nabla_h \mathbf{v}\,\mathrm{d}\mathbf{x} + \sum_{\kappa\in\mathcal{T}_h}\int_{\partial\kappa} \mathcal{H}(\mathbf{u}^{\mathrm{int}},\mathbf{u}^{\mathrm{ext}},\mathbf{n}_\kappa)\cdot\mathbf{v}^{\mathrm{int}}\,\mathrm{d}s \\
&\quad + \int_\Omega \mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h) : \nabla_h\mathbf{v}\,\mathrm{d}\mathbf{x} - \int_{\mathcal{F}_h} \{\!\{\mathcal{F}^v(\mathbf{u}_h,\nabla_h\mathbf{u}_h)\}\!\} : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}_h} \{\!\{G^\top(\mathbf{u}_h)\nabla_h\mathbf{v}\}\!\} : \underline{[\![\mathbf{u}_h]\!]}\,\mathrm{d}s + \int_{\mathcal{F}_h} \underline{\delta}(\mathbf{u}_h) : \underline{[\![\mathbf{v}]\!]}\,\mathrm{d}s, \\
\ell(\mathbf{v}) &:= \sum_{\kappa\in\mathcal{T}_h}\int_\kappa \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

where

$\{\!\{\cdot\}\!\}$: Average Operator $\qquad [\![\cdot]\!]$: Jump Operator

$$
\begin{aligned}
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\end{aligned}
$$

DGFEM

Find $\mathbf{u}_h \in S_{h,\vec{\mathbf{p}}}$ such that

$$
\mathcal{N}(\mathbf{u}_h,\mathbf{v}_h) = \ell(\mathbf{v}_h) \qquad \forall \mathbf{v}_h \in S_{h,\vec{\mathbf{p}}}.
$$

Penalty Parameter

Given $C_{\mathrm{ip}} > 0$, we define

$$\underline{\delta}(\mathbf{u}_h)|_f = C_{\mathrm{ip}} \frac{\mathrm{p}_f^2}{\mathrm{h}_f} \underline{[\![\mathbf{u}_h]\!]} \quad \text{for } f \in \mathcal{F}_h.$$

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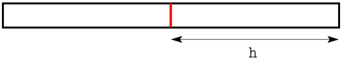
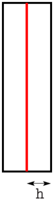
$$\mathrm{h}_f = \frac{\min\{\mathrm{vol}_d(\kappa_1), \mathrm{vol}_d(\kappa_2)\}}{\mathrm{vol}_{d-1}(f)}$$





Penalty Parameter

Given $C_{\rm ip} > 0$, we define

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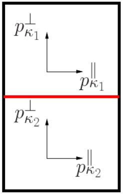
$$\mathrm{p}_f = \max\{p_{\kappa_1}^{\perp}, p_{\kappa_2}^{\perp}\}$$



Optimal order IPDGFEM

Three key ingredients:

1. Adjoint consistent imposition of the boundary terms present in $\mathcal{N}(\cdot,\cdot)$.

   Lu & Darmofal 2006, Hartmann 2007

2. Adjoint consistent reformulation of the target functional $J(\cdot)$.

   Harriman, Gavaghan, & Süli 2004, Hartmann 2007

3. Definition of the interior penalty terms.
   - Standard SIPG scheme

$$\underline{\delta}(\mathbf{u}_h) \equiv \underline{\delta}^{\mathrm{STSIPG}}(\mathbf{u}_h) = C_{\mathrm{ip}} \frac{\mathrm{p}_f^2}{\mathrm{h}_f} \underline{[\![\mathbf{u}_h]\!]}.$$

   - Modified SIPG scheme

$$\underline{\delta}(\mathbf{u}_h) \equiv \underline{\delta}^{\mathrm{SIPG}}(\mathbf{u}_h) = C_{\mathrm{ip}} \frac{\mathrm{p}_f^2}{\mathrm{h}_f} \{\!\{ G(\mathbf{u}_h) \}\!\} \underline{[\![\mathbf{u}_h]\!]}.$$

Outline

Adaptivity

- Goal:

$$|J(u) - J(u_h)| \leq \sum_{\kappa \in \mathcal{T}_h} |\eta_\kappa(u_h)| \leq \mathtt{Tol}.$$

- Automatic refinement algorithm:
  0. Start with initial (coarse) grid $\mathcal{T}_h^{(j=0)}$.
  1. Compute the numerical solution $u_h^{(j)}$ on $\mathcal{T}_h^{(j)}$.
  2. Compute the local error indicators η_κ.
  3. If $\sum_{\kappa \in \mathcal{T}_h} |\eta_\kappa| \leq \mathtt{Tol}$ → stop. Otherwise, .
  4. $j = j + 1$, and go to step (1).

R. Verfürth
A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques,
B.G. Teubner, Stuttgart, 1996.

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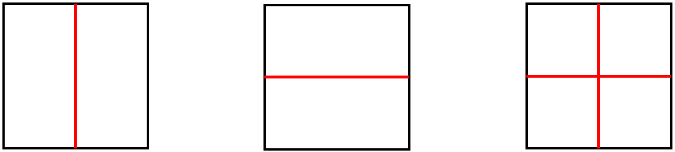
Adaptive Algorithm (h–Adaptivity)

- Elements marked for refinement/derefinement using the fixed–fraction strategy.
- Element refinements.



$$\mathcal{E}_1 \equiv \sum_{\kappa \in \mathcal{T}_{h,1}} |\eta_\kappa^{\text{new}}| \quad \mathcal{E}_2 \equiv \sum_{\kappa \in \mathcal{T}_{h,2}} |\eta_\kappa^{\text{new}}| \quad \mathcal{E}_3 \equiv \sum_{\kappa \in \mathcal{T}_{h,3}} |\eta_\kappa^{\text{new}}|$$

- Solve local primal and dual problems on elemental patches.
- Boundary data extracted from global primal and dual solutions.

Algorithm 1

Select optimal refinement

$$\max_{i=1,2,3}(|\eta_\kappa^{\text{old}}| - \mathcal{E}_i)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})).$$

Algorithm 2

- Prescribe an h–anisotropy parameter $\theta_h > 1$.
- When

$$\frac{\max_{i=1,2}(\mathcal{E}_i)}{\min_{i=1,2}(\mathcal{E}_i)} > \theta_h,$$

  perform refinement in direction with minimal $\mathcal{E}_i$, $i = 1, 2$.
- else perform isotropic h–refinement.

Adaptive Algorithm (h–Adaptivity)

Algorithm 1

Select optimal refinement

$$\max_{i=1,2,3}(|\eta_\kappa^{\text{old}}| - \mathcal{E}_i)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})).$$

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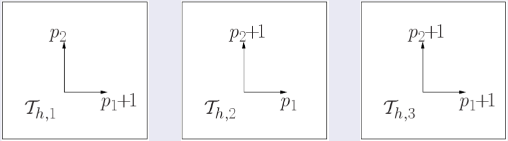
Anisotropic p–Refinement

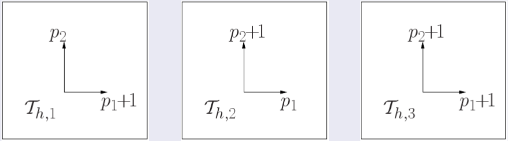
Local Problems



$$\mathcal{E}_1 \equiv |\eta_\kappa^{\text{new}}| \qquad \mathcal{E}_2 \equiv |\eta_\kappa^{\text{new}}| \qquad \mathcal{E}_3 \equiv |\eta_\kappa^{\text{new}}|$$

Anisotropic p–Refinement

Local Problems



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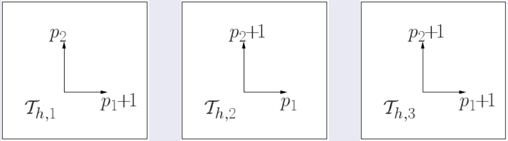
$$\max_{i=1,2,3} (|\eta_\kappa^{\text{old}}| - \mathcal{E}_i)/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})).$$

Anisotropic p–Refinement

Local Problems



$\mathcal{E}_1 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_2 \equiv |\eta_\kappa^{\text{new}}|$ $\quad$ $\mathcal{E}_3 \equiv |\eta_\kappa^{\text{new}}|$

Algorithm 2

- Prescribe a p–anisotropy parameter $\theta_p > 1$
- When

$$\frac{\max_{i=1,2}(\mathcal{E}_i/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})))}{\min_{i=1,2}(\mathcal{E}_i/(\#\text{dofs}(\mathcal{T}_{h,i}) - \#\text{dofs}(\mathcal{T}_{h,\kappa})))} > \theta_p,$$

  enrich in polynomial in the direction with minimal $\mathcal{E}_i$, $i = 1, 2$.
- else perform isotropic p–refinement.

Adaptive Algorithm (*hp*–Adaptivity)

- Elements marked for refinement/derefinement using the fixed–fraction strategy.
- Regularity estimation via truncated Legendre series expansions.
  Houston, Senior & Süli 2003, Houston & Süli 2005, Eibner & Melenk 2005.
- If both u and z are deemed to be non-smooth, apply anisotropic h–refinement.
- Else, perform anisotropic p–refinement

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$\mathrm{Ma} = 0.5$, $\mathrm{Re} = 5000$, $\alpha = 2^\circ$ and adiabatic wall condition.

Drag coefficients:

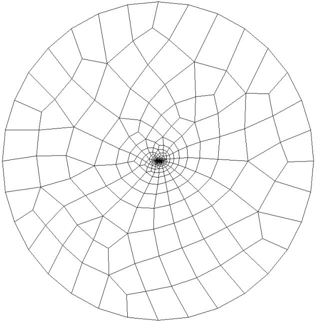
$$J_{c_{\mathrm{dp}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2}\int_S p\,(\mathbf{n}\cdot\psi_d)\mathrm{d}s, \qquad J_{c_{\mathrm{df}}}(\mathbf{u}) = \frac{2}{l\bar{\rho}|\bar{\mathbf{v}}|^2}\int_S (\boldsymbol{\tau}\,\mathbf{n})\cdot\psi_d\mathrm{d}s,$$

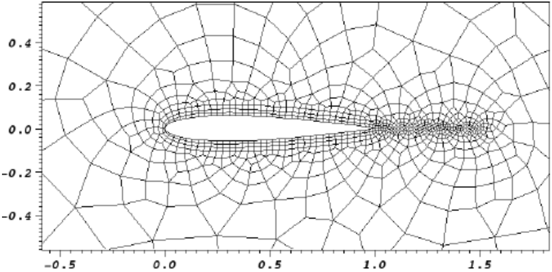
where

$$\psi_d = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

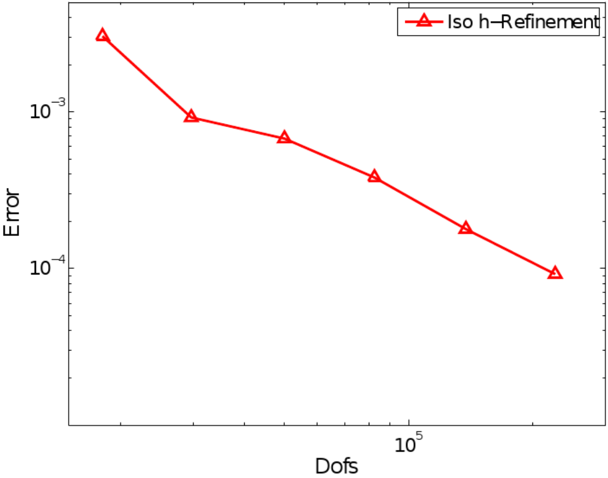
$$J_{c_{\mathrm{d}}}(\mathbf{u}) \approx 0.056084.$$

NACA0012 Airfoil – Unstructured





NACA0012 Airfoil – Unstructured



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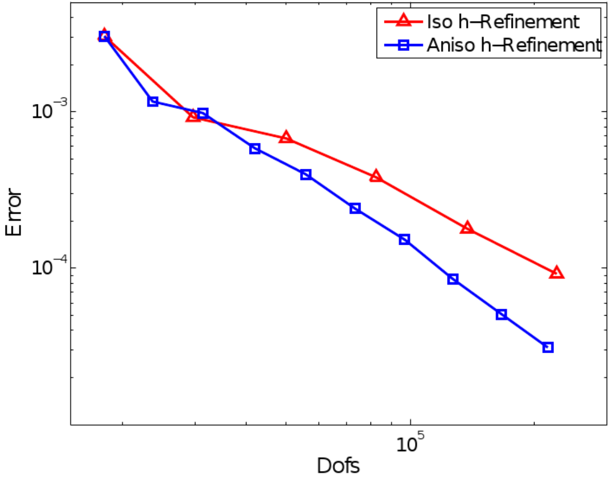
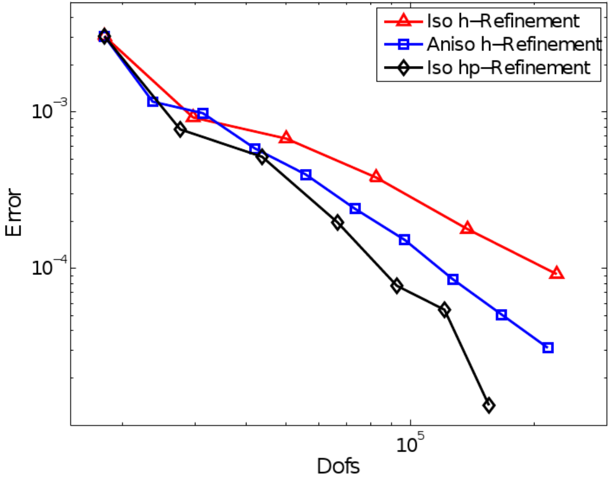


NACA0012 Airfoil – Unstructured



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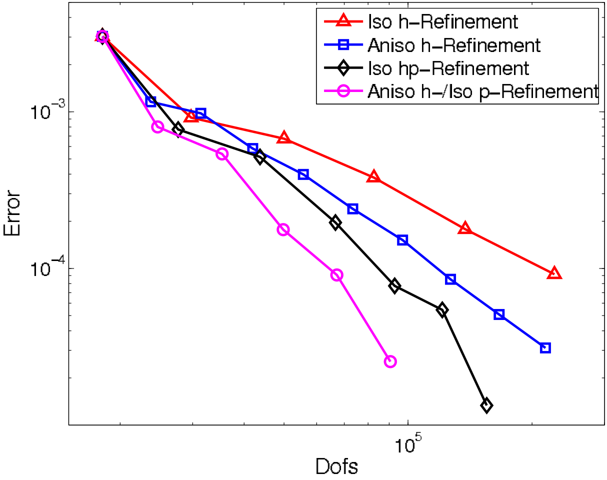

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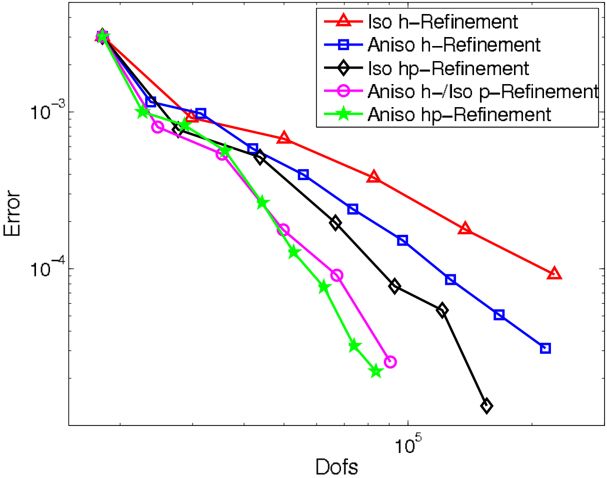

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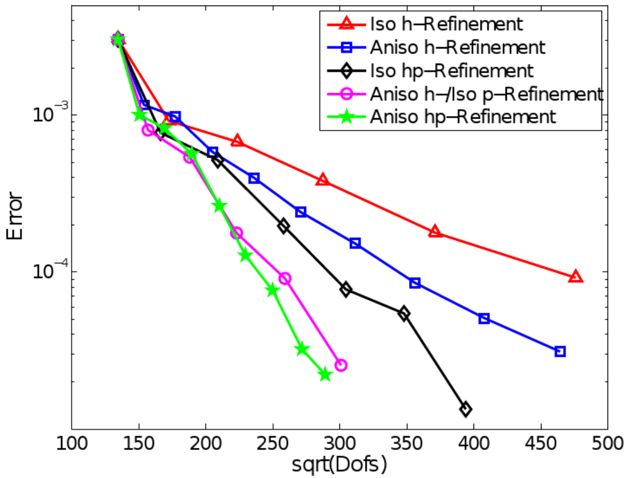

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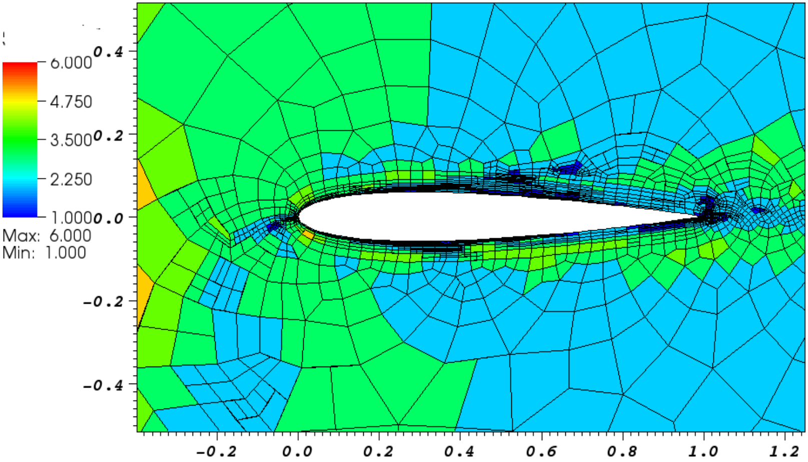
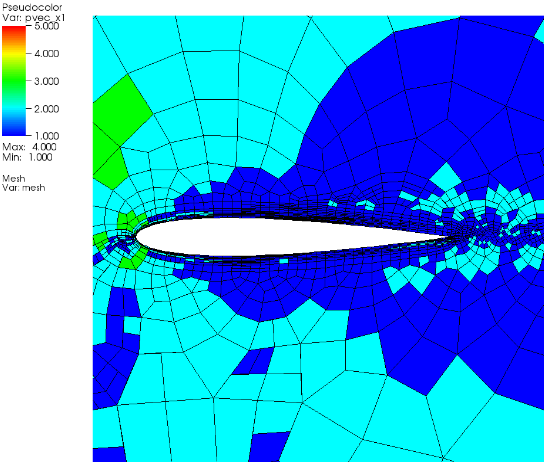


hp–mesh distribution after 6 adaptive (anisotropic *h*–/isotropic *p*–) refinements, with 2835 elements and 118520 degrees of freedom

NACA0012 Airfoil – Unstructured



hp/p_x–mesh distribution after 6 adaptive (anisotropic h–/anisotropic p–) refinements

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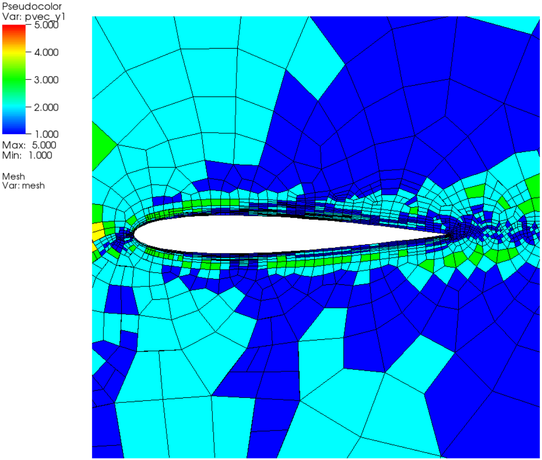
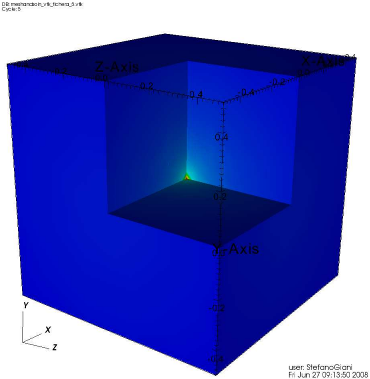


hp/p_y–mesh distribution after 6 adaptive (anisotropic h–/anisotropic p–) refinements

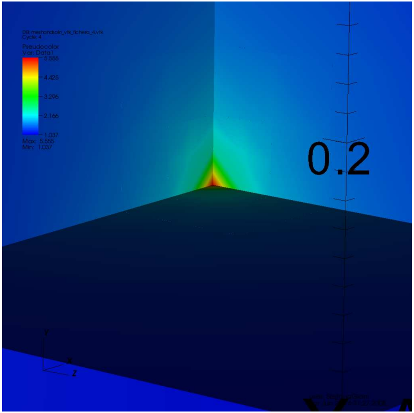
Outline

Fichera Corner - Isotropic *hp*–Refinement



$$\mathbf{u} = |\mathbf{x}|^{-1/4}\ , \quad \begin{cases} -\nabla \mathbf{u} = f, & \text{in } \Omega\ , \\ \mathbf{u} = \mathbf{u}_D, & \text{on } \partial\Omega\ . \end{cases}$$

Fichera Corner - Isotropic *hp*–Refinement



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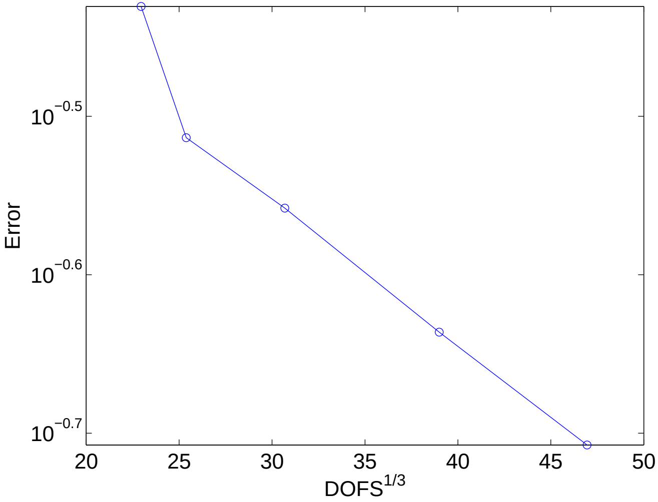

Fichera Corner - Isotropic *hp*–Refinement