High-Order/*hp*-Adaptive Multilevel Methods for Acoustics.

Stefano Giani

School of Mathematical Sciences, University of Nottingham, UK

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Outline

1. **Introduction**
2. **Limitation of standard FEMs**
3. **CFEDG**
   - An overview
   - Two–dimensional domain with micro-structures
4. **Numerics**
   - Acoustics problems
   - Goal-Oriented Error Estimator
5. **Conclusions**

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Adaptivity

- Goal:

$$\sum_{\kappa \in T_h} \tilde{\eta}_\kappa(u_h) \leq \texttt{Tol}.$$

- Automatic refinement algorithm:

  0. Start with initial (coarse) grid $\mathcal{T}_h^{(j=0)}$.
  1. Compute the numerical solution $u_h^{(j)}$ on $\mathcal{T}_h^{(j)}$.
  2. Compute the local error indicators $\tilde{\eta}_\kappa$.
  3. If $\sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa \leq \texttt{Tol} \rightarrow$ stop. Otherwise, adapt $S_{h,\vec{\mathbf{p}}}$.
  4. $j = j + 1$, and go to step (1).

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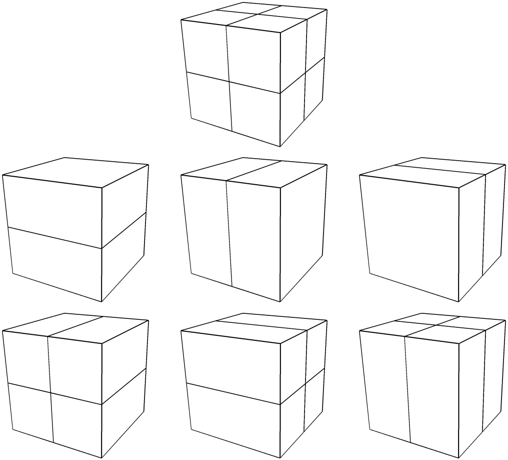
Isotropic refinement

- Isotropic *h*-refinement: The marked elements are split in smaller elements
- Isotropic *p*-refinement: The order of polynomials is increased on the marked elements
- Isotropic *hp*-refinement: The method automatically chooses to apply either isotropic *h*-refinement or isotropic *p*-refinement to each marked element

Anisotropic *h*-refinement



Anisotropic refinement

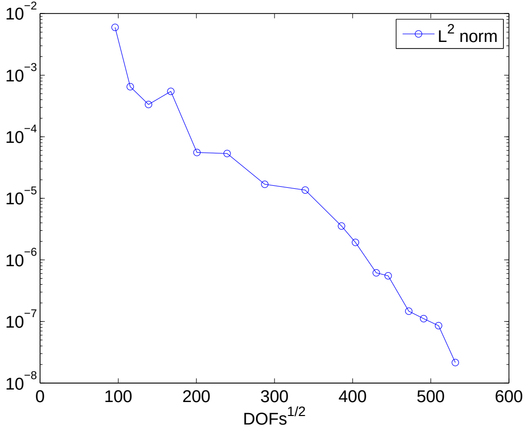
- Anisotropic *h*-refinement: The method automatically chooses to apply either isotropic or anisotropic *h*-refinement to each marked element
- Anisotropic *h*-isotropic *p*-refinement: The method automatically chooses to apply either isotropic/anisotropic *h*-refinement or isotropic *p*-refinement to each marked element
- Anisotropic *hp*-refinement: The method automatically chooses to apply either isotropic/anisotropic *h*-refinement or isotropic/anisotropic *p*-refinement to each marked element

Source Problem

$$\Omega \equiv [0,1]^2 \ , \quad \omega = 30000 \ , \quad c = 331.3 \text{ (air)} \ , \quad y_0 \equiv (0.51, 0.51)$$

$$-\Delta u - (\omega/c)^2 u = \delta_{y_0} \quad \text{in } \Omega, \qquad u = G_{y_0} \quad \text{on } \partial\Omega.$$

Initial mesh: 10×10, $p = 5$



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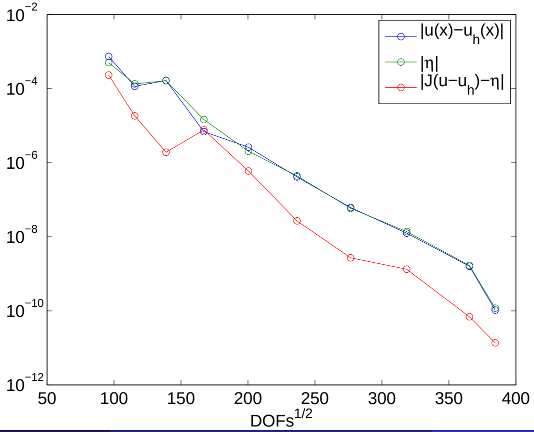
Initial mesh: 10×10, $p = 5$, $J(u) := \int_\Omega \delta_{x_0} u$, $x_0 \equiv (0.21, 0.21)$



A. Pietrzyk, "Accuracy of CAE predictions of NVH characteristics of a vehicle body in view of the dispersion in test", Computational modelling and analysis of vehicle body noise and vibration, University of Sussex, March 27-28 2012.

- They testes up to 34 identical (Volvo) vehicles.
- They measured the noise transfer function (NTF) - Acoustic response due to structural excitation.
- The standard deviation was 5dB.

Introduction

Fahy F. J.
Statistical energy analysis: a wolf in sheep's clothing?
Internoise '93, Leuven, 1993.

Reported a dispersion for beer cans over 10dB.

Kompella M. S., Bernhard B.J.
Measurement of statistical variation of structural-acoustic characteristics of automotive vehicles
SEA Noise and Vibr Conf., 1993.

Kompella M. S., Bernhard B.J.
Variational of structural-acoustic characteristics of automotive vehicles
Noise Control Eng. J., 44 (2) 1996.

Reported a dispersion of 10-20 dB.

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What is the main characteristic of an "efficient" finite element method (FEM)?
To be able to compute a good enough approximation of the solution using a very small number of degrees of freedom (DOFs)
What is limiting standard FEMs in acoustics?
It is the complexity of the geometry.

Outline

Ingredients:

1. a mesh $\mathcal{T}$
2. a finite element space $\mathcal{S}$ constructed on $\mathcal{T}$

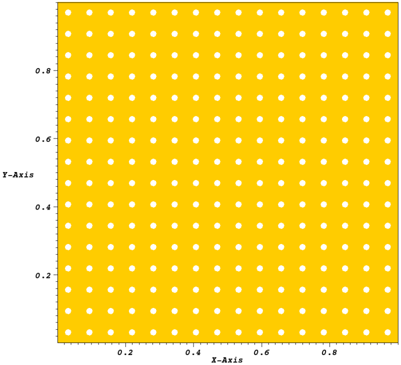
The standard way to construct a finite element space is to associate to each element in the mesh $\mathcal{T}$ a certain number of DOFs.

More element in the mesh $\mathcal{T} \Rightarrow$ more DOFs in the finite element space $\mathcal{S}$.

More DOFs $\Rightarrow$ more expensive problem to solve.

Complicated Geometry

Plate with 256 circular holes



a lot of elements are necessary to describe the geometry and this implies a lot of DOFs.

Thousands of elements $\Rightarrow$ thousands of DOFs.

A Simple Though

DOFs are used to approximate the solution.

So, DOFs should be placed only where it is useful to improve the accuracy of the approximated solution.

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Solution ...

The construction of the finite element space $\mathcal{S}$ should be independent on mesh.

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CFEDG (composite finite element discontinuous Galerkin) method:

P. Antonietti, S.G. and P. Houston
hp-version composite discontinuous Galerkin methods for elliptic problems on complicated domains
SISC, submitted.

It is an extension of CFE for continuous Galerkin:

W. Hackbusch and S.A. Sauter
Composite finite elements for the approximation of PDEs on domains with complicated micro-structures
Numer. Math., 75, 447–472, 1997.

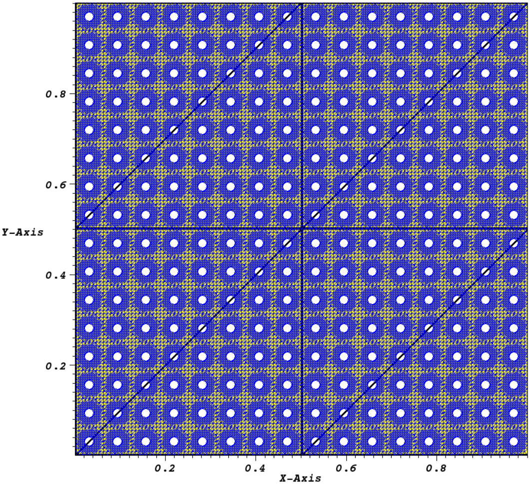
W. Hackbusch and S.A. Sauter
Composite finite elements for problems containing small geometric details. Part II: Implementation and numerical results
Comput. Visual Sci., 1, 15–25, 1997.

Two levels

Two meshes:

- Mesh $\mathcal{T}_{h_\ell}$ is a partition of the domain Ω and describes all the details in the domain.
- A coarser mesh $\mathcal{T}_{\mathrm{CFE}}$ which is to coarse to describe the details in the domain Ω.

Two levels

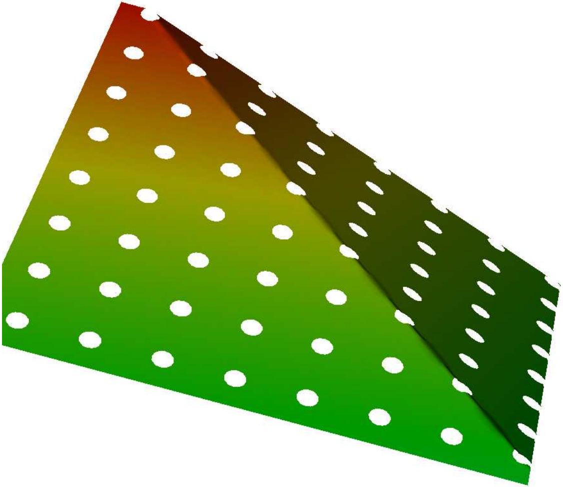


Two levels

Transferring information from the fine level to the coarse level:

1. The geometrical details are not “stored” in the coarse mesh $\mathcal{T}_{\mathrm{CFE}}$ but in the finite element basis functions on the coarse level.
2. The mesh $\mathcal{T}_{\mathrm{CFE}}$ and the correspondent finite element space $V(\mathcal{T}_{\mathrm{CFE}}, p)$ are used to set the size and the sparsity of the linear system.

Basis Functions



Outline

Problem

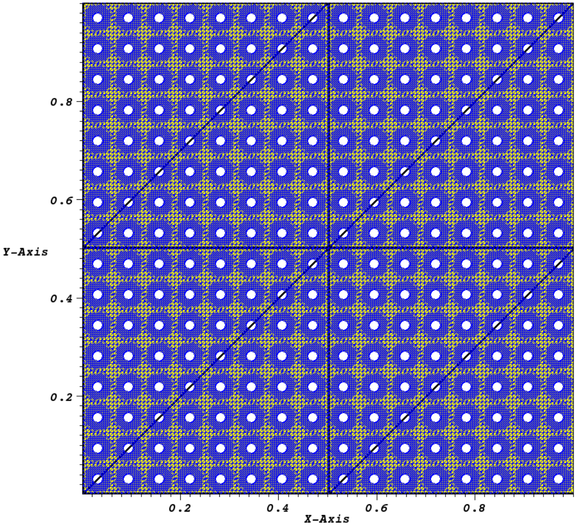
$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega, \\
u &= g \text{ on } \partial\Omega.
\end{aligned}
$$

Solution:

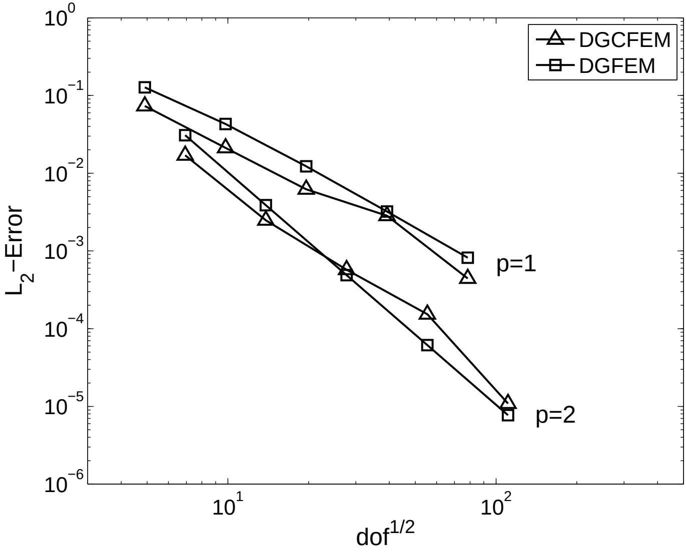
$$u = \sin(\pi x)\cos(\pi y).$$

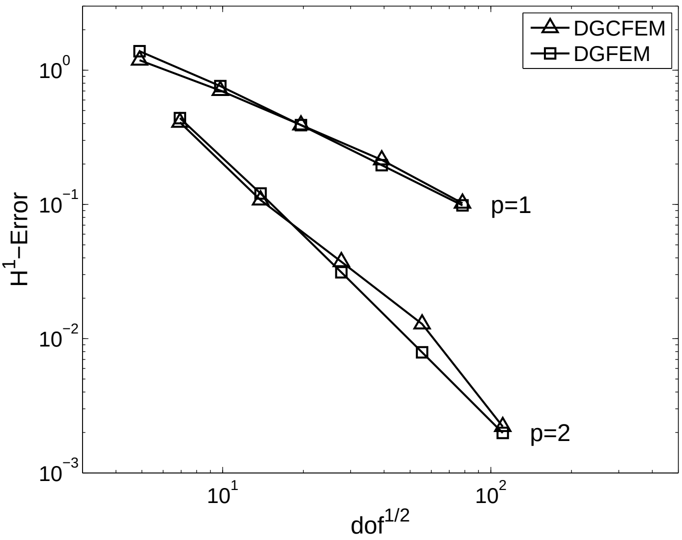
Ω has micro-structures.

Domain



Convergence in L^2




DGCFEM
DGFEM
10^0
10^{-1}
10^{-2}
10^{-3}
H^1–Error
10^1
10^2
$dof^{1/2}$
p=1
p=2

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\begin{aligned}
-\Delta u - (k/c)^2 u &= 1 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega.
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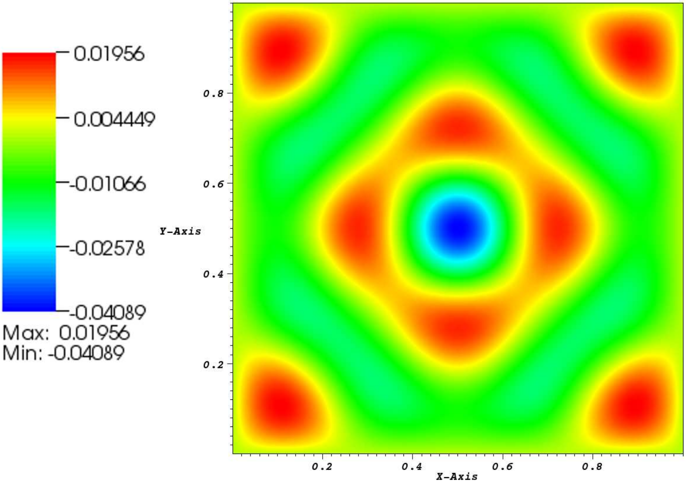
With $k = 6000$, $c = 331.3$

Importance of small features



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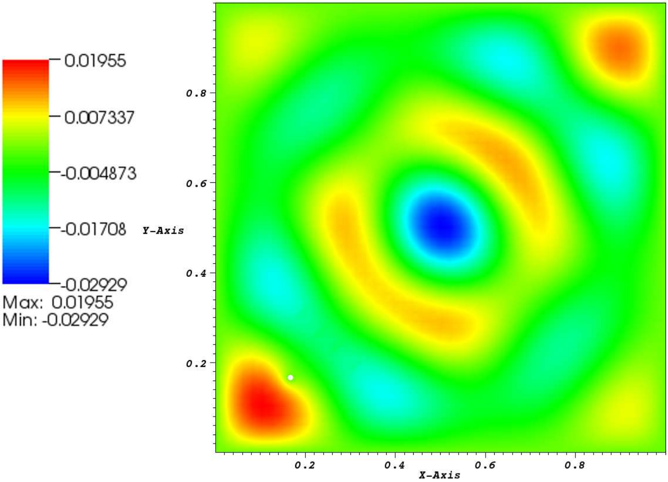
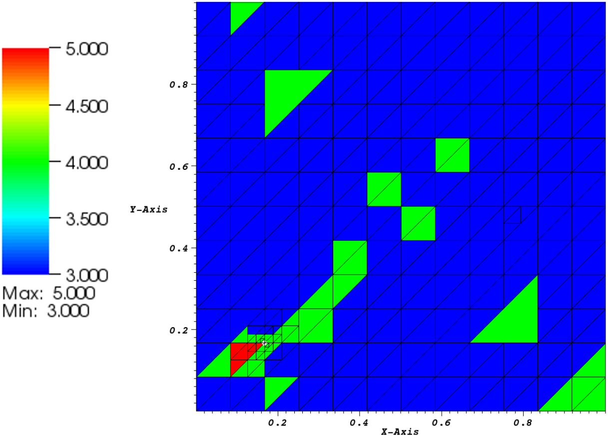


Hole of diameter 0.01.



DOFs: 3597, Elements: 327, $\|u - u_h\|_0 = 4.5128E - 005$

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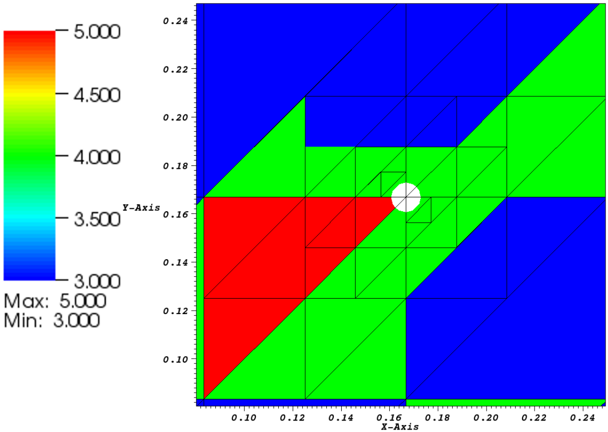

DOFs: 3597, Elements: 327, $\|u - u_h\|_0 = 4.5128E - 005$

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Goal-Oriented Error Estimator

Primal linear problem:

$$a(u,v) \;=\; F(v)\ .$$

Dual linear problem:

$$a(v,z) \;=\; J(v)\ , \quad J(v) \;:=\; \int_\Omega \delta_{x_0} v\ .$$

Goal oriented error estimator: $u_{hp} \in S_{h,\vec{\mathbf{p}}}$

$$J(u) - J(u_{hp}) \;=\; a(u - u_{hp}, z) \;=\; F(z) - a(u_{hp}, z)\ .$$

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$$J(u) - J(u_{hp}) \;=\; a(u-u_{hp},z) \;=\; F(z) - a(u_{hp},z) \;=\; \sum_{\kappa\in\mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp},z)\;,$$

$$|J(u) - J(u_{hp})| \;\leq\; \sum_{\kappa\in\mathcal{T}_h} |\tilde{\eta}_\kappa(u_{hp},z)|\;.$$

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Goal oriented error estimator: $u_{hp} \in S_{h,\vec{\mathbf{p}}}$, $\tilde{z}_{hp} \in \tilde{S}_{h,\tilde{\mathbf{p}}+\mathbf{1}}$

$$J(u)-J(u_{hp}) \;\approx\; a(u-u_{hp},\tilde{z}_{hp}) \;=\; F(z)-a(u_{hp},\tilde{z}_{hp}) \;=\; \sum_{\kappa\in\mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp},\tilde{z}_{hp})\;,$$

$$|J(u)-J(u_{hp})| \;\lesssim\; \sum_{\kappa\in\mathcal{T}_h} |\tilde{\eta}_\kappa(u_{hp},\tilde{z}_{hp})|\;.$$

Pros & Cons

Cons:

1. **Expensive:** a dual problem should be solved

Pros:

1. **Flexible:** the quantity of interest can be chosen
2. **Sharp Bound:** the upper bound has constant 1

$$|J(u) - J(u_{hp})| \lesssim \sum_{\kappa \in \mathcal{T}_h} |\tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp})| \ .$$

3. **Accurate:** it is possible to compute an accurate estimation of the error in the quantity of interest

$$J(u) - J(u_{hp}) \approx \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp}) \ ,$$

4. **Improved Accuracy:** it is possible to improve the accuracy of the quantity of interest

$$J(u) \approx J(u_{hp}) + \sum_{\kappa \in \mathcal{T}_h} \tilde{\eta}_\kappa(u_{hp}, \tilde{z}_{hp}) \ ,$$

$$
\begin{aligned}
-\Delta u - (k/c)^2 u &= 1 \text{ in } \Omega, \\
u &= 0 \text{ on } \partial\Omega.
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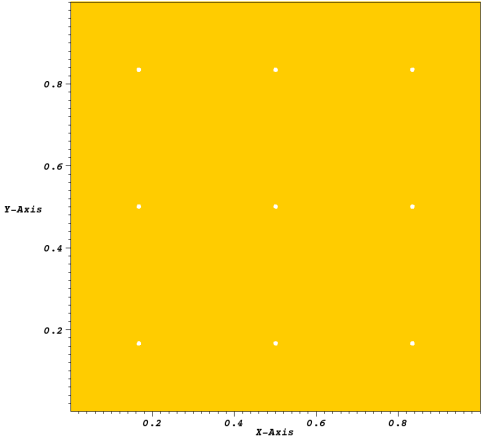
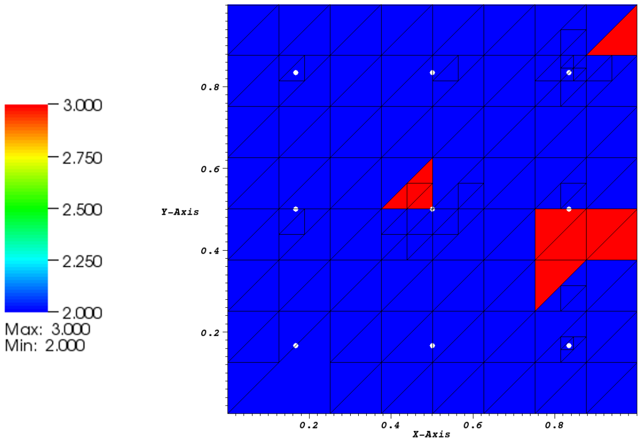
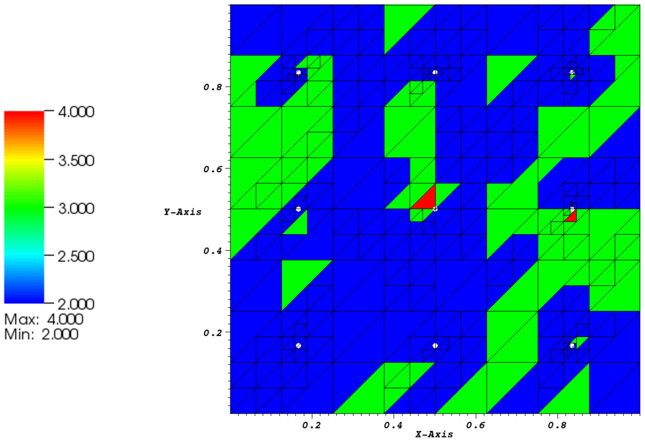
With $k = 6000$, $c = 331.3$. The domain has 9 small holes.
The point of interest is (0.71 0.69).

Goal-oriented





Mesh 6 - DOFs: 1110, Error: -5.4524 dB.



Mesh 9 - DOFs: 3334, Error: -8.6696 dB.

Outline

Conclusions

1. CFEDG can solve a problem with complicated geometry with few DOFs.
2. *hp*-adaptivity can introduce new DOFs in an efficient way only where it is important to improve the accuracy of the solution.
3. You have full control on the number of DOFs in your system.
4. CFEDG can be used as a preconditioner as well.
5. CFEDG is memory efficient