Convergence of adaptive FEM for elliptic eigenvalue problems

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ZSS 08

Outline

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   - Efficiency
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Model Problem

Let Ω be a polygonal domain bounded in $\mathbb{R}^2$
Problem: seek eigenpairs (λ,u) of the problem

$$\begin{cases} -\nabla \cdot \mathcal{A}(\nabla u) &= \lambda \mathcal{B} u \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega. \end{cases}$$

$$\underline{a} \le \xi^T \mathcal{A}(x) \xi \le \overline{a} \quad \text{for all} \quad \xi \in \mathbb{R}^2 \text{ with } |\xi| = 1 \text{ and for all} \quad x \in \Omega,$$

$$\underline{b} \le \mathcal{B}(x) \le \overline{b} \quad \text{for all} \quad x \in \Omega.$$

Variational Problem: seek eigenpairs $(\lambda, u) \in \mathbb{R} \times H^1_0(\Omega)$ such that

$$a(u, v) \; = \; \lambda (u, v)_{0,\mathcal{B}\Omega} \quad \text{for all } v \in H^1_0(\Omega),$$

where

$$a(u, v) \; := \int_\Omega \mathcal{A} \nabla u \cdot \nabla v \, dx, \quad |||u|||_\Omega := a(u, u)^{1/2} \, .$$

$$(u, v)_{0,\mathcal{B},\Omega} \; := \int_\Omega \mathcal{B} \, u \, v \, dx.$$

Meshes and Discrete Problems

Define:

- $\mathcal{T}_n$ conforming and shape regular triangulation of Ω
- $\mathcal{F}_n$ is the set of the edges of the triangles in the interior of $\mathcal{T}_n$,
- V_n space of piecewise linear functions over $\mathcal{T}_n$.

Problem: seek eigenpairs $(\lambda_n, u_n) \in \mathbb{R} \times V_n$ such that

$$a(u_n, v_n) = \lambda_n (u_n, v_n)_{0,\mathcal{B},\Omega} \quad \text{for all } v_n \in V_n.$$

Reliability for Eigenfunctions

Theorem (Reliability for eigenfunctions)

Let (λ, u) *be a simple eigenvalue of the continuous problem and let* (λ_n, u_n) *be the correspondent computed eigenpair. Then we have for* $e_n = u - u_n$ *that*

$$|||e_n|||_\Omega \leq C\ \eta_n\ +\ G_n,$$

where

$$G_n = \frac{1}{2}(\lambda + \lambda_n)\frac{(e_n, e_n)_{0,\mathcal{B},\Omega}}{|||e_n|||_\Omega}.$$

[R. Verfürth (1996). *A Review of Posteriori Error Estimation and Adaptive Mesh Refinement Techniques*. Wiley-Teubner]

Residual

Definition (Jump)

$$[g]_f(x) := \left(\lim_{\substack{\tilde{x}\in\tau_1(f)\\ \tilde{x}\to x}} g(\tilde{x}) \; - \lim_{\substack{\tilde{x}\in\tau_2(f)\\ \tilde{x}\to x}} g(\tilde{x}) \right), \quad \text{with } x \in f.$$

Definition (Residual)

$$R_I(u,\lambda)(x) := (\nabla \cdot \mathcal{A}\nabla u \; + \; \lambda\mathcal{B}u)(x), \quad \text{with } x \in int(\tau), \quad \tau \in \mathcal{T}_n,$$

$$R_F(u)(x) := \left[\vec{n}_f \cdot \mathcal{A}\nabla u\right]_f(x), \quad \text{with } x \in int(f), \quad f \in \mathcal{F}_n.$$

$$\eta_n := \left\{ \sum_{\tau\in\mathcal{T}_n} H_\tau^2 \| R_I(u_n,\lambda_n)\|_{0,\tau}^2 \; + \; \sum_{f\in\mathcal{F}_n} H_f \| R_F(u_n)\|_{0,f}^2 \right\}^{1/2},$$

Reliability for Eigenfunctions

Theorem (Reliability for eigenfunctions)

Let (λ, u) be a simple eigenvalue of the continuous problem and let (λ_n, u_n) be the correspondent computed eigenpair. Then we have for $e_n = u - u_n$ that

$$|||e_n|||_\Omega \leq C\,\eta_n \; + \; G_n,$$

where

$$G_n = \frac{1}{2}(\lambda + \lambda_n)\frac{(e_n, e_n)_{0,\mathcal{B},\Omega}}{|||e_n|||_\Omega}.$$

Reliability for Eigenvalues

Theorem (Reliability for eigenvalues)

Let (λ, u) be a simple eigenvalue of the continuous problem and let (λ_n, u_n) be the correspondent computed eigenpair. Then we have for $e_n = u - u_n$ that

$$|\lambda_n - \lambda| \leq C' \ \eta_n^2 \ + \ G'_n,$$

where

$$G'_n = \eta_n \frac{1}{2}(\lambda + \lambda_n) \frac{(e_n, e_n)_{0,\mathcal{B},\Omega}}{|||e_n|||_\Omega} \ + \ \frac{1}{2}(\lambda - \lambda_n)(e_n, e_n)_{0,\mathcal{B},\Omega}.$$

Efficiency

Theorem (Efficiency)

Let (λ, u) *be a simple eigenpair of the continuous problem and let* (λ_n, u_n) *be the corresponding computed eigenpairs. Then we have that the global residual estimator is bounded by the energy norm of the error as:*

$$\eta_n \leq C' \,|||e_n|||_\Omega \;+\; \|H_\tau(\lambda_n u_n - \lambda u)\|_{0,\mathcal{B},\Omega}.$$

Marking Strategy 1

Set the parameter $0 < \theta < 1$:
mark the sides in a minimal subset $\hat{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that

$$\left(\sum_{\tau \in \hat{\mathcal{T}}_n} \eta_{\tau,n}^2 \right)^{1/2} \geq \theta \eta_n,$$

where

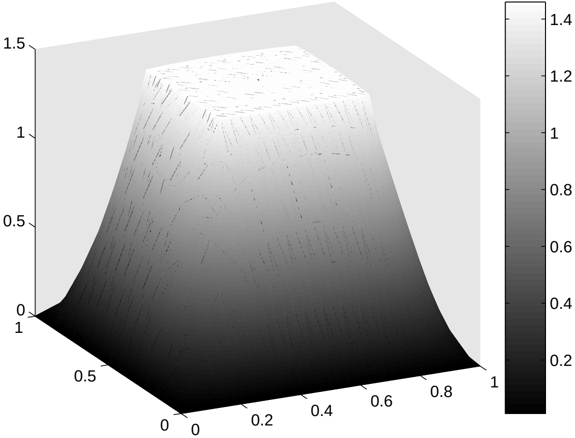
$$\eta_{\tau,n}^2 := H_\tau^2 \| R_I(u_n, \lambda_n) \|_{0,\tau}^2 + \sum_{f \in \partial\tau} \frac{1}{2} H_f \| R_F(u_n) \|_{0,f}^2.$$

Mesh Adaptivity Algorithm

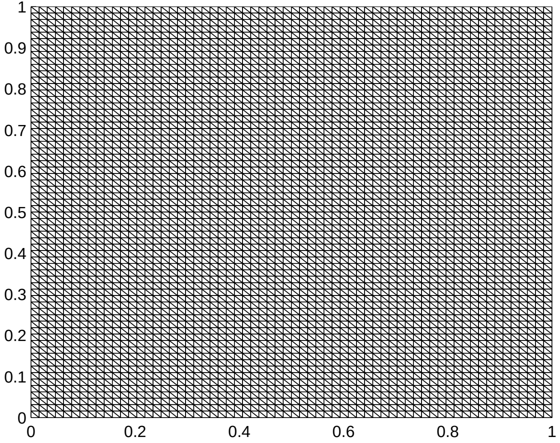
1. Set tol, $0 < \theta < 1$ and $\mathcal{T}_0$
2. Loop
3. Compute (λ_n, u_n) on $\mathcal{T}_n$
4. Marking strategy 1
5. Refine the mesh
6. End Loop if $\eta_n \leq \mathrm{tol}$

Solution

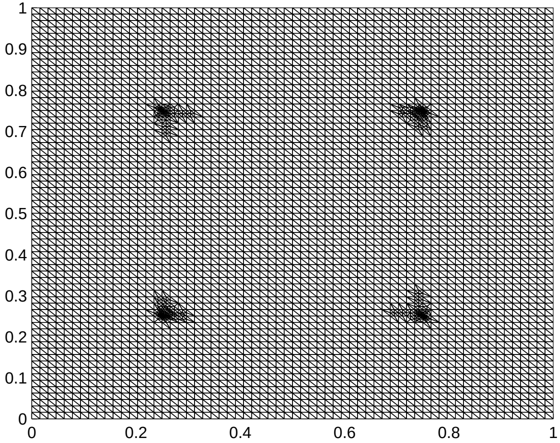
$$\mathcal{B} = 1, \quad \mathcal{A} = \{1, 10000\}.$$



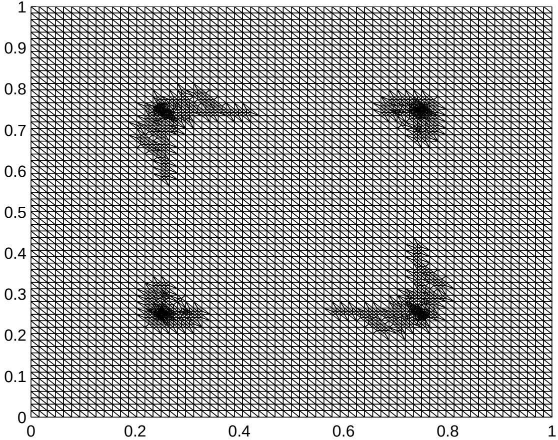
Uniform Mesh



Refined Mesh



Refined Mesh



Numerics

$$\mathcal{B} = 1, \quad \mathcal{A} = \{1, 10000\}.$$

Uniform			Adaptive					
			$\theta = 0.5$			$\theta = 0.8$		
$\lvert\lambda - \lambda_n\rvert$	N	n	$\lvert\lambda - \lambda_n\rvert$	N	n	$\lvert\lambda - \lambda_n\rvert$	N	n
1.1071	81	1	1.1071	81	1	1.1071	81	1
0.3521	289	2	0.2766	673	5	0.2244	799	3
0.1168	1089	3	0.0948	2080	8	0.0990	2235	4
0.0399	4225	4	0.0315	6039	11	0.0180	12375	6
0.0136	16641	5	0.0148	12607	13	0.0065	29148	7
0.0042	66049	6	0.0038	37126	16	0.0020	65387	8

Table: Comparison between uniform refinement and the adaptive method for the smallest eigenvalue of the generic elliptic eigenvalue problem with discontinuous coefficients.

Oscillations

[P. Morin, R. H. Nochetto and K. G. Siebert (2000). Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal. 38*]

Oscillations:

$$osc(v_n, \mathcal{T}_n) := \left(\sum_{\tau \in \mathcal{T}_n} \| H_\tau (v_n - P_n v_n) \|_\tau^2 \right)^{1/2},$$

where $(P_n v_n)|_\tau := \frac{1}{|\tau|} \int_\tau v_n$

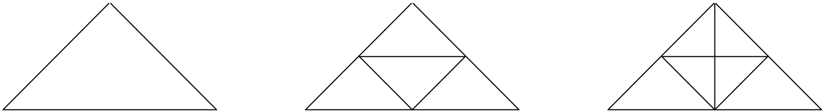
Marking Strategy 2

Set the parameter $0 < \tilde{\theta} < 1$:
mark the sides in a minimal subset $\tilde{\mathcal{T}}_n$ of $\mathcal{T}_n$ such that

$$osc(u_n, \tilde{\mathcal{T}}_n) \geq \tilde{\theta}\; osc(u_n, \mathcal{T}_n).$$

Then we take the union of $\hat{\mathcal{T}}_n \cup \tilde{\mathcal{T}}_n$ and we refine all the elements in the union.

Bisection5



1. Split any edge
2. Split one of the new edge

PROS:

1. A new node on each edge
2. A new node in the interior of the element

Mesh Adaptivity Algorithm

1. Require $0 < \theta < 1$, $0 < \tilde{\theta} < 1$ and $\mathcal{T}_0$
2. Loop
3. Compute (λ_n, u_n) on $\mathcal{T}_n$
4. Marking strategy 1
5. Marking strategy 2
6. Refine the mesh
7. End Loop

Convergence for Linear Problems

$$-\nabla \cdot (\mathcal{A} \nabla u) = f \, .$$

[P. Morin, R. H. Nochetto and K. G. Siebert (2000). Data oscillation and convergence of adaptive FEM. *SIAM J. Numer. Anal. 38*]

- osc decreases refining the mesh.

$$\mathrm{osc}(f, \mathcal{T}_{n+1}) \leq \alpha \, \mathrm{osc}(f, \mathcal{T}_n), \quad \text{with } \alpha < 1.$$

- The decay of osc trigs the convergence in the energy norm.

$$a(u_n, u_n)^{1/2} \leq C_0 \, p^n, \quad \text{with } p < 1.$$

Convergence for Eigenvalue Problems

$$-\nabla \cdot (\mathcal{A} \nabla u) \; = \; \mathcal{B} \, \lambda \, u \, .$$

[S. Giani and I. G. Graham (2007). A convergent adaptive method for elliptic eigenvalue problems. *submitted for publication*]

- The decay of osc depends on the behavior of the error.

$$\mathrm{osc}(u_{n+1}, \mathcal{T}_{n+1}) \; \leq \; \tilde{\alpha} \, \mathrm{osc}(u_n, \mathcal{T}_n) \; + \; C'' \, a(u - u_n, u - u_n)^{1/2}.$$

- The decay of the error depends on the behavior of osc.

$$|||u - u_{n+1}|||_{\Omega}^2 \; \leq \; \beta_n \, |||u - u_n|||_{\Omega}^2 \; + \; C''' \, \mathrm{osc}(\lambda_n u_n, \mathcal{T}_n)^2.$$

- The error and the quantity osc are linked together.
  If H_0 is small enough:

$$|||u - u_n|||_{\Omega} \; \leq \; C_0 \, p^n, \quad |\lambda - \lambda_n| \; \leq \; C_1 \, p^{2n},$$

$$\mathrm{osc}(u_n, \mathcal{T}_n) \; \leq \; C_3 \, p^n, \quad \text{with } p \; < \; 1.$$